# EUCLID

ISAAC BARROW

# In Newton's shadow

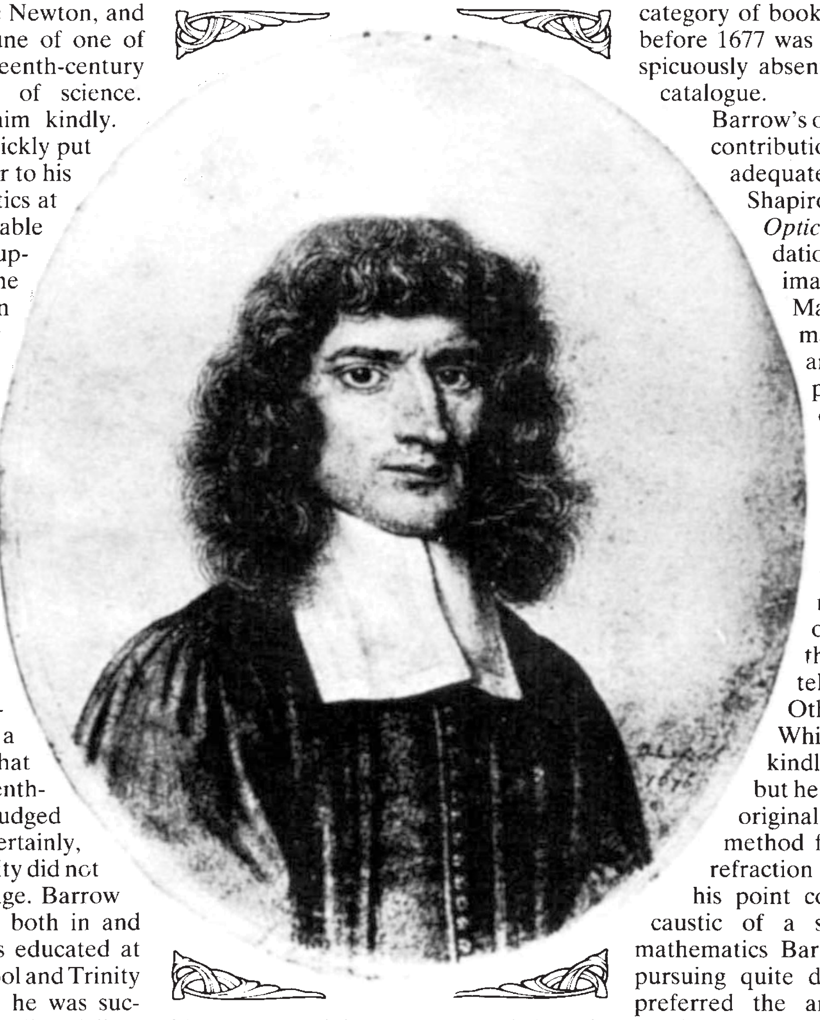
*Willem Hackmann*

**Before Newton. The Life and Times of Isaac Barrow.** Edited by Mordechai Feingold. *Cambridge University Press: 1990. Pp. 380. £35, $49.50.*

When one thinks of Isaac Barrow one immediately thinks of Isaac Newton, and that has been the misfortune of one of the most prominent seventeenth-century mathematicians and men of science. History has not treated him kindly. Barrow's reputation was quickly put in the shade by the successor to his Lucasian chair of mathematics at Cambridge, the incomparable Newton, whom Barrow supported successfully for the post. His reward has been that he has remained a rather shadowy figure which this biography, consisting of a collection of chapters by Barrovian authorities, attempts to illuminate.

According to an earlier biographer, Barrow was known to his contemporaries for his slovenly ways, his excessively long sermons, and his great learning. He is best remembered for his mathematical and optical writings. History always has difficulty in judging a polymath and that is what Barrow was in seventeenth-century terms. Others have judged him an ambitious careerist. Certainly, as this biography shows, ability did not get one far without patronage. Barrow was lucky with his patrons both in and outside Cambridge. He was educated at Charterhouse, Felstead School and Trinity College, Cambridge, where he was successively a pensioner (1643), college fellow (1649), and Master (1673). His first attempt at the Regius professorship of Greek failed for politico-religious reasons (Cambridge was in the throes of the Civil War), but his second attempt (1660) at the restoration of the monarchy was successful. Subsequently he became Gresham professor of geometry in London (1662), thereby supplementing his meagre Cambridge income, the first Lucasian professor of mathematics (1663), Royal Chaplain to Charles II in London (1669), and finally Vice-Chancellor of Cambridge (1675).

Barrow's misfortune was, as Feingold points out in his introductory chapter 'Isaac Barrow: divine, scholar, mathematician', that he was active at the turning point of classical culture in the mid-seventeenth century — a learned culture that became virtually obsolete in Barrow's lifetime, when a more narrow scholarly professionalism started to creep in. Past historians of science, too, have largely ignored Barrow in their quest for the 'giants' of the scientific revolution, and his name is linked to no discovery. Thus, Barrow became chiefly remembered as the mentor of Newton, but even this relationship has never been clarified. That Newton was Barrow's pupil at Trinity is a myth. His name does not appear in Newton's surviving papers, and there is no evidence to suppose that Newton's early mathematical or optical discoveries were due to Barrow. These aspects are discussed in both Feingold's introductory chapter and in John Gascoine's chapter, entitled 'Isaac Barrow's academic milieu: Interregnum and Restoration Cambridge'. Both chapters also give a good insight into mid-seventeenth-century university education at Cambridge.



The picture shows Isaac Barrow in 1676 (courtesy of the National Portrait Gallery).

Insight into the relationship between Barrow and Newton is also somewhat unexpectedly given in the final chapter in this volume in which Feingold describes Barrow's library. As his first biographer, Abraham Hill, wrote, the estate left by Barrow "was books". Of his library of 1,100 volumes, predictably for the period more than 80 per cent were in Greek or Latin, and the remainder in English, French, Italian, Hebrew and Arabic. The books were almost evenly distributed among Barrow's three main interests: theology, science (about half of these were mathematical and the remainder were devoted to medicine and philosophy) and the humanities. Newton had free access to Barrow's library and this is the reason why he was not compelled, as Feingold suggests, to acquire a large number of books before Barrow's death in 1677. The largest single category of books purchased by Newton before 1677 was alchemy, a subject conspicuously absent from Barrow's library catalogue.

Barrow's optical and mathematical contributions have been very adequately dealt with in Alan Shapiro's chapter, entitled 'The *Optical Lectures* and the foundations of the theory of optical imagery', and in Michael Mahony's 'Barrow's mathematics: between ancients and moderns'. Shapiro points out that Barrow completed a major phase of the Keplerian revolution in geometrical optics by creating a mathematical theory of optical imagery. The weakness was that he did not deal with applied optics and such topics as the eye and vision, the telescope and microscope. Other biographers, notably Whiteside, have been less kindly about Barrow's optics, but he does agree that his most original contributions were his method for finding the point of refraction at a plane interface and his point construction of the diacaustic of a spherical interface. In mathematics Barrow and Newton were pursuing quite different paths. Barrow preferred the analysis of geometrical configurations; Newton's fluxions (his form of calculus) clearly lay in the algebraic tradition.

Barrow emerges from these pages a scholar in the humanist tradition (VIDE Anthony Grafton's brief chapter, 'Barrow as a scholar'), a sincere man with a gentle temper (see Irène Simon's equally short chapter, 'The preacher') and, to give the final word to the editor of this volume, a man whose integrity is the recurring motif of his life. ☐

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## To cite this version:

Marco Panza. Isaac Barrow and the Bounds of Geometry. Patricia Radelet De-Grave. Liber Amicorum Jean Dhombres, Brepols, pp.365-411, 2008. <hal-00456290>

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Submitted on 13 Feb 2010

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# Isaac Barrow and the Bounds of Geometry[*]

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December 16, 2007

During the 17th century, mathematics changed deeply. As it has often been remarked, this went together with a new methodological or, rather, a new philosophical attitude. Nevertheless, in the 17th century, philosophy of mathematics was not just a marginal effect or an implicit and, of course, partial cause of a broad transformation of mathematics itself. It was also a domain of research and the subject of controversies and intellectual disputes; in short, an autonomous discipline[1]. Of course, the internal evolution of such a discipline could not remain completely unaffected by the contemporary evolution of mathematics. Still, many of the topics that constituted its specific content were intrinsically dependent on traditional quarrels and quite old questions. One of these topics was concerned with the mutual relations and the opposite virtues of arithmetic and geometry. Though inherited from the traditional distinction of quantities in the opposite genera of numbers (or discrete quantities) and magnitudes (or continuous quantities), this question appeared in a new light because of the impressive disclosure of new algebraic techniques to be applied both to numbers and magnitudes.

Isaac Barrow and John Wallis, the most prominent and innovative British mathematicians of their time, devoted much of their intellectual energy to such a topic, arguing for two diametrically opposite views. Barrow's *Lectiones Mathematicæ*[2] and Wallis's *Mathesis uni-*

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[*] The material discussed in the present paper was first presented at the HOPOS conference held in Paris, on June 2006, during a joint talk with Antoni Malet. I thank Antoni Malet himself and David Rabouin for having drawn my attention to this material and for the many instructive discussions of it and related subjects. I also thank Michel Blay, Sara Confalonieri, Massimo Galuzzi, Daniel Garber, Niccolò, Guicciardini, Doug Jesseph, Vincent Jullien, Susan Klaiber, Paolo Mancosu, Sébastien Maronne, and Theodora Seal, for valuable comments on my talk and earlier versions of my paper. Many thanks also to Luigi Maierù for several very stimulating discussions about Barrow and Wallis, among other things.

[1] This view is, at least implicitly, endorsed in [30], [23], [31], and [17].

[2] Cf. [9] and [6]. Though published only in 1683, six years after Barrow's death, this treatise includes the lectures given by him at the University of Cambridge, as Lucasian professor of mathematics in 1664-1666 (the crucial years in the mathematical schooling of Newton: cf. [27], vol. I, and [28]). In 1734, an English translation of it, due to J. Kirkby, was published: cf. [7]. I quote from this translation. A new edition of the original Latin text is available in [8], pp. 1-378. Concerning the passages which are relevant for my purpose, Kirkby's translation is accurate enough. Thus, I'll quote it, though pointing out some typographical errors, and suggesting how to correct it in the few cases where it seems to me to differ considerably from the original.

*versalis* and *Algebra*[3] were quite elaborate treatises aiming to defend (according to different methodological strategies) two opposite theses: the foundational superiority of geometry over arithmetic and of arithmetic over geometry, respectively.

My aim is to discuss two examples that Barrow relies on in his lecture III. His conception would of course deserve more attention and a much more detailed analysis[4]. Here I shall limit myself to a very specific point, which I chose because it is very close to a topic discussed by Jean in one of his papers which I particularly favor for both scientific and personal reasons[5].

## 1 Barrow's thesis about arithmetic and geometry

The following quotation, taken from the very beginning of lecture III, should suffice to point out the thesis defended by Barrow in this lecture[6]:

> [...] I affirmed the Whole of Mathematics to be in some sort contained and circumscribed within the Bounds of Geometry. And indeed concerning the other Sciences then mentioned, I imagine no Body will much deny, but they are conveniently enough reduced to Geometry. But now I have brought a harder Task upon myself, to shew allowable and fit Causes for expunging *Arithmetic* out of the List of Mathematical Sciences, and as it were degrading the most noble Science from the Degree which it has been so long possessed of. [...] Be it far from me to take away or seclude a Science so excellent and profitable that of Numbers from the Mathematics. I will rather restore it into its lawful Place, as being removed out of its proper Seat, and ingraff and unite it again into its native Geometry, the Stock from whence it has been plucked. [...] For I am convinced that Number[7] really differs nothing from what is called Continued Quantity, but is only formed to express and declare it; and consequently that Arithmetic and Geometry are not conversant about different Matters, but do both equally demonstrate Properties

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[3] Cf. [41], [42] and [43].

[4] For a general description and an assessment of the whole treatise, cf. [18], pp. 181-202. The matter has also been treated by: C. Sasaki—in the context of a study of the acceptance of the theory of proportion in the 16th and 17th centuries [cf. [34]]—, H. Pycior—in the context of a reconstruction of the historical setting for Berkeley's views about mathematics [cf. [32], pp. 274-277]—, L. Maierù—in the context of a general reconstruction of the mathematical works of Barrow and Wallis [cf. [19]]—, K. Hill—in the context of a study of Wallis' and Barrow's conceptions about the composition of continua [cf. [15]]—, A. Malet—in the context of a comparison between Barrow's and Wallis attitudes toward the method of indivisibles [cf. [21]] and in that of a reconstruction of Barrow's program of mathematization of natural philosophy [cf. [22]]—, and D. Jesseph—in the context of the controversy between Hobbes and Wallis' on the quadrature of circle.

[5] Cf. [11].

[6] Cf. [7], pp. 29-31.

[7] Kirkby's translation omits here a relevant parenthesis: "(illum saltem quem Mathematicus contemplatur)". One could include it in the main statement thus: "For I am convinced that Number (at least in so far as it is considered by a mathematician) really differs nothing..."

> common to one and the same Subject; and very many, and very great Improvement will appear to be derived from hence upon the Republic of Mathematics.
>
> I. For first of all it appears that there is neither any general Axiom nor particular Conclusion agreeing with Geometry (which respects Magnitudes not taken absolutely, but in Comparison with one another according to certain Proportions of Equality or Inequality, *i. e.* as they are capable of being designed and compared together by Measures) but what by the same Reason also agrees with Arithmetic: And on the other hand that nothing can be affirmed, concluded, or demonstrated, concerning Numbers which may not in like manner be accommodated to Magnitudes; whence accrues a remarkable Light and vast Improvement to both Sciences; the Cause being removed of superfluously repeating and demonstrating Theorems by Nature altogether the same, also of often solving the same Problems.[...]
>
> II. This Coalition of Numbers and Magnitudes being admitted, a plentiful Accession accrues to each Discipline. For it will be a very easy thing to discover and demonstrate very many Theorems concerning Numbers by the Assistance of Geometry, which, by keeping within the common Limits of Arithmetic would scarcely, if at all, be capable either of Investigation or Demonstration: Also very many Things may be more briefly and clearly found out and demonstrated from hence. And reciprocally, the Ratio's or Reasons of Numbers being well understood will communicate not a little to the more evident Explication and strong Confirmation of many Geometrical Theorems.

To sum it up in Mahoney's words[8], Barrow's thesis is that "the domain of arithmetic [...][is] subsumed under that of geometry", or, as Jesseph says[9], that "numbers, far from being self-subsistent objects, are mere symbols whose content derives from their application to continuous geometric magnitude".

If we considered it in the light of the internal evolution of mathematics, this proposal—and specially Barrow's promise of a "great Improvement [...] upon the Republic of Mathematics" that should have been derived from the envisioned reduction of arithmetic to geometry—would inevitably appear as completely out of place. It was made in spring 1664[10], more than seventy years after Viète's axiomatic definition of an algebraic formalism for any sort of quantity[11], almost thirty years after Descartes interpretation of this same formalism on any sort of magnitude[12], and on the immediate eve of Newton's results that brought him to the theory of fluxions[13]. These results did certainly not provide any argument to deny that number “is only formed to express and declare” geometric magnitudes, that it merely results from a way to speak of them. But, in the light of them, the essential question for the evolution of mathematics could no more be that of the intrinsic nature of numbers and their relations with magnitudes. Questions connected with the development of an appropriate formalism to be used for treating both numbers and magnitudes became much more relevant, indeed.

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[8] Cf. [18], p. 188.

[9] Cf. [17], p. 33.

[10] Cf. [13], p. 68.

[11] Cf. [37], ch. 2.

[12] Cf. [10], pp. 297-298.

[13] Cf. [27], vol. I, parts I and II, and [28]. It is quite probable that Barrow's lectures were published from a manuscript that he had given to Collins in March 1668 [cf. [13], pp. 68-79; specially, p. 69.]. At that date,

Still, the right perspective in which Barrow’s proposals should be read is not that of the evolution of mathematics. Barrow was rather looking for a way to found the new mathematical results on the secure roots of the classical tradition. This is a twofold philosophical requirement: it is both a requirement of foundation and a requirement of tradition. Many historical reasons could explain it. I merely suggest that it was such a double philosophical requirement that commanded Barrow’s *Lectiones Mathematicæ*, and that it was for this reason that this work is so profoundly involved with topics, arguments and rhetorical figures derived from the Aristotelian philosophical repertory. This suggestion has a historiographical implication: the program exposed in this work should not be understood as it were intended to promote some sort of mathematical progress, and should not be judged on the basis of the natural criteria used to judge a mathematical program. I guess that, *mutatis mutandis*, this is also the case of the opposite program exposed in Wallis’ *Mathesis universalis* and *Algebra*.

The previous long passage that I have quoted from the third of Barrow’s *Mathematical Lectures* is followed by two examples that Barrow introduced without any foreword: “We will illustrate the Matter with an Example or two”, he simply wrote[14]. Barrow provided, in each case, what he considered to be a geometric incontestable proof of an arithmetical theorem, and pretended that no equally exact or clear proof of these statements could be obtained by means of purely arithmetic arguments[15].

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Barrow was certainly already aware of the results obtained by Newton between 1664 and 1666. And, though, at that time, Newton had not yet coined the term “fluxion” and did not have such a general concept as that which he designated with this term in the *De methodis* (composed in 1671), these results should have shown, to any attentive interpreter, that the mathematical properties of quantities depended much less on their specific nature than on their possible operational relations.

[14] Cf. [7], p. 31.

[15] The term “arithmetic” certainly did not have, in the middle of the 17th century, a fixed and univocal meaning. It was used both in a strict sense, to refer to that part of mathematics that dealt with numbers through the basic operations of addition, subtraction, multiplication, division and extractions of roots defined on them, and in a broader sense, to refer to what was also termed, more precisely, “universal arithmetic”. The term “algebra” was even more ambiguous. Very often, historians use it today—when speaking of early modern mathematics—to refer to the same matter that the term “universal arithmetic” denoted in early modern age, and this was also a common use at that time [about the notion of algebra in early modern age, cf. [29]]. It follows that there is room, when early modern mathematics is concerned, to understand the terms “arithmetic” and “algebra” as synonymous. It seems to me that this understanding does not apply to the case under consideration, however. I rather suggest that in his third lecture, Barrow was using the former term in the first of the two previous senses. At the end of his second lecture, he was, moreover, quite explicit in arguing that algebra should be understood as “a Part or Species of Logic”, that is, as a “certain Manner of using Reason”, and that, so understood, it could apply both to geometry and arithmetic [cf. [7], p. 28]. It seems thus that for

The first example concerns a theorem that is actually very hard to prove in purely arithmetical terms. Nevertheless, Barrow's geometric proof depends on the supposition of a sort of solidarity between arithmetic and geometry that—despite being clearly justified—could have been understood, and actually had been understood by Wallis, as a symptom of a possible reduction of geometric results to arithmetical ones, rather than *viceversa*. The theorem involved in the second example admits, instead, a number of simple purely arithmetical proofs, some of which were certainly known or could have been known (and understood) by Barrow. Hence, were it judged on a purely mathematical basis, Barrow's claim would appear in both cases as highly questionable. If we are not ready to admit that Barrow's examples have—and also had, at Barrow's time—no argumentative force, and simply reduce to rhetorical devices, we have, thus, to offer a different interpretation of them.

I suggest that what Barrow was arguing for, through these examples, was not the greater deductive power of geometry over arithmetic, but rather the foundational priority of a form of infinity, quite naturally involved in geometric arguments, over another form of infinity, usually involved, instead, in arithmetical arguments. In other words, I suggest that what Barrow was implicitly maintaining, when he asserted that the theorems under consideration did not admit a satisfactory purely arithmetical proof, was that such a proof would in any case be grounded on some infinitary arguments that should have not been admitted on a purely arithmetic basis, but could have been justified only if infinity were understood geometrically. Under this interpretation, his arguments depend on an hidden premise (and, in a sense, finally reduce to an implicit assertion of this same premise): infinity can only be conceived, correctly, as the potential infinity of the possible divisions of a given finite magnitude; an infinite sum of terms is correctly conceivable only if these terms represent the infinitely many parts in which an already given finite magnitude can be divided.

Though Barrow's general thesis about the subsumption of the domain of arithmetic under that of geometry, constituted, as Mahoney himself remarks[16], a "radical departure from tradition", the more local thesis that he was arguing for, by relying on his two examples, seems thus to be nothing but a consequence of a well-known Aristotelian view, that Barrow explicitly endorsed in his ninth lecture[17]:

> Indeed we can approach nothing with Sense, but as it is terminated [...]. Neither we can conceive any Magnitude in Thought, at least distinctly, but only as it is contained and comprehended within some Limits. We may indeed have a confused Imagination of some Magnitude in Power, suppose a Right Line, as *Aristotle* said,

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Barrow, the problem of the relations between arithmetic and geometry was merely that of the relations between arguments or methods concerned with numbers (independently from the way they were understood) and the basic operations on them, and arguments or methods concerned with geometric magnitudes. According to him, the fact that these arguments or methods could or could not take an "algebraic" form—that is, use symbols for species (or *logisice speciosa*)—was apparently not relevant for this question.

[16] Cf. [18], p. 186. Cf. also [15], p. 173.

[17] Cf. [7], pp. 142-143.

> infinite; *i. e.* produced or encreased at pleasure, or to a greater Distance than any Distance assigned, which consequently was denominate infinite. Nor is the Philosopher's Definition of Infinite improper, though very subtle; *That is infinite, from which if one take any Quantity, something still may be assumed beyond*[18]. But this nothing else but to consider many Lines successively, according to a certain arbitrary and indeterminate Power, and not one real Indeterminate Line distinctly; which according to the Philosopher again we cannot comprehend in the Mind. [...]. The same is of an *infinite Subduction* or *Subdivision*, by which nothing else is understood, but that a Subtraction or Division may be continued at pleasure, without being brought to an Impossibility of proceeding farther. Again, *Aristotle* speaking of Mathematicians says, *Mathematicians do neither want nor use infinite Magnitude, but take as much as they please, when they are minded to terminate it*[19].

## 2 Parabola and square roots: Barrow's inversion of an argument of Wallis

Here is the first of Barrow's examples[20]:

> It is an Arithmetical Theorem, that *the Sum of an infinite (*or *indefinite) Series of Numbers increasing from Nothing to a certain Term, which is the greatest according to the Ratio of the square Roots of Numbers continually exceeding one another by Unity, (i.e. as* $0$, $1$, $\sqrt{2}$, $\sqrt{3}$, *&c.* ad infinitum*) is subsesquialter the Sum of as many equal to the said greatest Term*; Which Theorem I am of Opinion can never be exactly demonstrated by any Method in Arithmetic itself: but it is plainly deduced from Geometry.

In modern terms, Barrow's theorem could be stated as follows:

$$\lim_{n\to\infty} \frac{\sum\limits_{i=0}^{n} \sqrt{i}}{(n+1)\sqrt{n}} = \frac{2}{3}. \tag{1}$$

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[18] Barrow refers to *Physics* III. 9. The reference to Bekker's text is rather 207$a$, 7-8. Here is Hardie's and Gaye's translation, revised by J. Barnes [cf. [4], vol. I, p. 352]: "Thus something is infinite if, taking it quantity by quantity, we can always take something outside."

[19] Barrow refers to *Physics* III. 11. The reference to Bekker's text is rather 207$b$, 27-31. Here is Hardie's and Gaye's translation revised by J. Barnes [cf. [4], vol. I, p. 354]: "Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that a finite straight line may be produced as far as they wish."

[20] Cf. [7], p. 31.

Hence, his last claim is, strictly speaking, false: this theorem can be proved by purely arithmetical means. Still, this is quite hard to do, and Barrow can certainly not be blamed for not having grasped this possibility. A way to get a proof is to rely on a beautiful result obtained by Srinivasa Ramanujan[21] that provides a general expression for the sum $\sum_{i=0}^{n} \sqrt{i}$, for any value of $n$. I expose this proof in the annex 1.

If no general expression for the sum $\sum_{i=0}^{n} \sqrt{i}$ is known, there is no way to prove Barrow's theorem in arithmetical terms. The more natural and easy non arithmetical way to prove it is through an argument based on an appropriate integration. In the annex 2, I show how this argument works.

The geometric proof suggested by Barrow is, in a sense, an informal version of such an argument. Here is his how he presents it[22]:

> For if the Diameter of any Parabola be conceived to be divided indefinitely into many equal Parts, then the Right Lines which are ordinately applied to the Diameter, through the Points of the Divisions[23], will proceed in the same Ratio, as is shewn in Geometry: But the Parabola which is constituted of these, whether Right Lines or Parallelograms, is there also demonstrated to be Subsesquialter to the Parallelogram, upon the same Base and the same Height, or which is the same thing, to the Sum composed of as many Right Lines or Parallelograms equal to the greatest: From whence, the Agreement of Arithmetic with Geometry being supposed which we desire to advance, it plainly follows, that a Series of Numbers of this sort is Subsesquialter the Sum of as many equal to the greatest.

Supposing that $\omega$ is an infinite (integer) number, the theorem we have stated before in the form (1) could be stated as follows

$$\frac{\sum_{i=0}^{\omega} \sqrt{i}}{(\omega + 1)\sqrt{\omega}} = \frac{2}{3}, \tag{2}$$

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[21] Cf. [33]. Ramanujan's short note contains a more general result than the one that this proof appeals to. This last result is stated in the first part of section 1: less that ten lines of text.

[22] Cf. [7], p. 31.

[23] Kirkby's translation omits here a parenthesis: (vel parallelogramma que altra illis insistentia)". Barrow seems not to be willing, here, to take a definite decision about the dimension of the indivisibles he is using. This is, in fact, an irrelevant question, for the purpose of the argument he is exposing (hence, I shall not insist on this point either, in my commentaries). Later, namely in lecture IX, he comes back on the question by discussing both the notion of indivisible and the grounds of the method of indivisibles, and arguing that the former is rightful insofar as indivisibles are taken not to be homogenous with the magnitudes they are the indivisibles of and these magnitudes are not supposed to be composed by them, but the latter is correct insofar as indivisibles are replaced by infinitesimals, which are, instead, homogenous parts of the relevant magnitudes. Cf. on this matter [21], pp. 75-81.

which seems to me to be closer to Barrow's conceptions.

To prove that it is so, Barrow suggested inverting the type of arguments that Wallis had relied on, in his *Arithmetica infinitorum*[24], in order to justify his method of quadrature. In the case of the parabola[25], Wallis' argument depended on the preliminary admission of the arithmetic equality

$$\frac{\sum\limits_{i=0}^{h} i^2}{(h+1)\,h^2} = \frac{1}{3} + \frac{1}{6h}. \tag{3}$$

By supposing that the (finite) sums entering the ratio figuring in the left-hand side of this equality are replaced by "infinite series", he got, then,

$$\frac{\sum\limits_{i=0}^{\omega} i^2}{(\omega+1)\,\omega^2} = \frac{1}{3}, \tag{4}$$

where $\omega$ is an infinite number[26]. This was, for him, a sufficient basis for concluding that[27] the ratio between the curvilinear triangle $\mathsf{OMA}$ and the parallelogram $\mathsf{OPMA}$ is that of 1 to 3, supposing that $\mathsf{OM}$ is an arc of parabola of vertex $\mathsf{O}$, $\mathsf{OP}$ is a portion of the axis of this same parabola, and $\mathsf{OA}$ is tangent to $\mathsf{OM}$ at $\mathsf{O}$.

The reason he advanced for justifying this conclusion was that the segments parallel to $\mathsf{OP}$, drawn from $\mathsf{OA}$ up to $\mathsf{OM}$ (and thus their projections over $\mathsf{OP}$), are to the segments parallel to $\mathsf{OA}$, drawn from $\mathsf{OP}$ up to $\mathsf{OM}$ "in ratione duplicata" (and the segments parallel to $\mathsf{OA}$, drawn from $\mathsf{OP}$ up to $\mathsf{OM}$—and thus their projections over $\mathsf{OA}$—are to the segments parallel to $\mathsf{OP}$, drawn from $\mathsf{OA}$ up to $\mathsf{OM}$ "in subduplicata ratione"), and that the curvilinear triangle $\mathsf{OMA}$ "consists of" the former, whereas the parallelogram $\mathsf{OPMA}$ "consists of all the segments like $\mathsf{OM}$, equal to the greatest of them."

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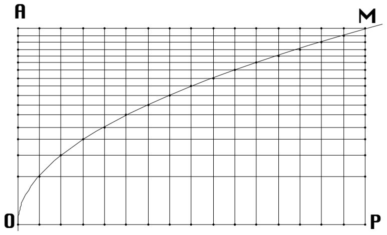
[24] Cf. [40] and [28], ch. 1. Similar methods were also imagined and employed by Fermat, Mengoli, Pascal, and Roberval: cf.: [44], p. 319; [5], pp. 150-156; [20], p. 36; and [24], pp. 259 and 265-268.

[25] Cf. [40], propp. XIX-XXIII, pp. 15-18.

[26] As a matter of fact, Wallis did not write any equality like (4). He simply claimed [cf. [40], prop. XXI, p. 17] that: "Si pro ponatur series infinita Quantitatum in dupplicata ratione arithmetice-proportionaliū [. . . ] continue crescentium, a puncto seu 0 inchoatarum; erit illa ad seriem totidem maximaæ æqualium, ut 1 ad 3."

Considering the simpler case of the ratio $\frac{\sum_{i=0}^{h} i}{(h+1)h}$ (which is easily proved to be equal to $\frac{1}{2}$, for any value of $h$), Wallis was a little bit more explicit about the nature of his sums [cf. [40], prop. II, p. 2] : "Si sumatur series Quantitatum Arithmeticè proportionalium [. . . ] continuè crescentium, a puncto vel 0 inchoatarum, & numero quidem vel finitarum vel infinitarum (nulla enim discriminis causa erit,) erit illa ad seriem totidem maximaæ æqualium, ut 1 ad 2." It seems thus that Wallis was willing to avoid any explicit discussion about the difference between the finitary and the infinitary cases of his equalities, by presenting the latter as quite natural (that is, unproblematic) extensions of the former. My mere replacement of "$h$" with "$\omega$" is just intended to reflect this attitude.

[27] Cf. [40], propp. XXIII, pp. 17-18.



Wallis' method had thus three main components. In the case of the parabola, these are the following: the establishment of the equality (3) and the infinitary extension of this result enabling one to get the equality (4); the identification of the curvilinear triangle OPM and the parallelogram OPMA—at least when their ratio was concerned—with appropriate infinite aggregates of parallel segments; the assimilation of the ratio of these aggregates of segments with the ratio of appropriate series of numbers[28].

Barrow's argument shares the last two components of Wallis' method, but, by inverting the direction of deduction, replaces the first one with the admission of a previous quadrature of the parabola. Though he did not mention his sources for this last result, he would have certainly mentioned the proposition 24 of Archimedes' *Quadrature of the parabola*, if the question had been asked of him. Two years after the publication of Barrow's *Mathematical Lectures*, in 1685, the English (and first) edition of Wallis's *Algebra* was published, and there, the latter came back to his method of quadrature[29], by ascribing to the same Archimedes the crucial result (3), that he identified with the content of the proposition 10 of the treatise *On Spiral*. Moreover, he appropriately accommodated Archimedes' proof of such a proposition so to get a perfectly rigorous proof, by *reducio ad absurdum*, of this arithmetical result, and maintained that analogous proofs could also be given for the corresponding results entering the other applications of his method of quadrature contained in the *Arithmetica infinitorum*.

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[28] As it is clearly shown by the quotations included in the previous footnote (26), Wallis claimed that the equality (4), as well as the corresponding equality $\frac{\sum_{i=0}^{\omega} i}{(\omega+1)\omega} = \frac{1}{2}$, hold for any series of quantities in double or arithmetical proportion, respectively. This is quite obvious, since, for any exponent $n$ and any quantity $\alpha$, one has:

$$\frac{\sum_{i=0}^{h} (i\alpha)^n}{(h+1)(h\alpha)^n} = \frac{\alpha^n \sum_{i=0}^{h} i^n}{\alpha^n (h+1) h^n} = \frac{\sum_{i=0}^{h} i^n}{(h+1) h^n},$$

and the problem of determining the first of these ratios reduces to that of calculating the last one. The crucial point in Wallis' argument is not concerned, however, with the simple and quite obvious equality of a ratio of series of segments (that is, magnitudes) and a ratio of series of numbers, but rather with the assimilation of these ratios with a ratio of infinite aggregates of segments, each of which constitutes a bidimensional figure. As a matter of fact, this is a sort of analogy between an arithmetical and a geometrical relation.

[29] Cf. [42], pp. 298-304.

Though this last claim is highly questionable (at least if possibility is understood as an effective one, with respect to the actual capacities of early-modern mathematicians), Wallis's appeal to the authority of Archimedes would have been, in itself, sufficient for denying that Barrow's argument could provide a geometric justification for the equality (3) that were somehow more appropriate than any arithmetical justification.

But, as a matter of fact, this is not what Barrow claimed. His point was subtler. He rather claimed that the interpretation of a ratio of arithmetical sums as a ratio of geometric figures is appropriate to show that the equality (4) is equivalent to the equality (2). In other words, Barrow's point seems to be the following: once this interpretation is admitted and the quadrature of the parabola is obtained, the passage to arithmetic can be realized in two trivially equivalent ways: either by remarking that the arithmetical ratio which corresponds to the ratio of the curvilinear triangle $\mathsf{OMA}$ and the parallelogram $\mathsf{OPMA}$—according to a partition of these figures in their components parallel to the axis $\mathsf{OP}$—is $\frac{1}{3}$, or by remarking that the arithmetical ratio corresponding to the ratio of the curvilinear triangle $\mathsf{OPM}$ and the parallelogram $\mathsf{OPMA}$ —according to a partition of these figures in their components parallel to the tangent $\mathsf{OA}$—is $\frac{2}{3}$. The triviality of this equivalence is the gain obtained by the involvement of geometry that Barrow seems to underline.

This is an effective gain, indeed: so effective that Wallis himself had surreptitiously taken advantage of it, by claiming, in the *Arithmetica infinitorum*[30], that, provided that the "complement of the semi-parabola" (that is, $\mathsf{OMA}$) is to the parallelogram $\mathsf{OPMA}$ as 1 to 3, "consequently, the semi-parabola itself is to the same parallelogram as 2 to 3."

But is this gain enough to support the thesis of a subsumption of the domain of arithmetic under that of geometry? Three points should be made here.

Firstly, one should observe that, without the help of some version of the *calculus*—which Barrow did not know in 1664 and that could hardly be understood as a natural extension of classical geometry—a similar gain could not be but a local one, since it is quite hard to generalize Archimedes' quadrature of the parabola with the mere tools of classical geometry. This simple remark seems to me strong enough to show that Barrow's argument was, from the point of view of the internal evolution of mathematics, completely out of place.

Secondly, it is easy to remark that, in getting the equality (2) by relying on the quadrature of the parabola, rather than on an infinitary extension of an appropriate finite equality analogous to the equality (3)—that is, of an equality equivalent to the equality (27), providing Ramanujan's expression for the sum $\sum\limits_{i=0}^{n} \sqrt{i}$, for any value of $n$—Barrow's argument does not provide any clue as to such a finite equality. Certainly, as already said, no one could blame Barrow for not having got Ramanujan's expression. Still, his very argument shows that geometry—at least if it is used the way he suggested—is not powerful enough to lead to any arithmetical result that one could be willing to obtain.

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[30] Cf. [40], propp. XXIII, pp. 17.

Thirdly, and mainly, I think: one cannot forget that Barrow's argument shares the second and third main components of Wallis' method. So the question is obvious: how could one pretend to have offered an argument for the subsumption of the domain of arithmetic under that of geometry, by relying on an argument based on the assimilation of the ratio of some infinite aggregates of segments with the ratio of some series of numbers? This assimilation could, indeed, be read both to show that geometry provides a model for the mutual relations between the terms of an arithmetical sum, and to show that the relevant features of this model are also the features of an arithmetical ratio. This is a general problem concerning Barrow's thesis: the claim that arithmetic is nothing but an appropriate language to speak about (geometric) magnitudes seems to be nothing but the reverse of the claim that arithmetic is the pure and more abstract expression of the relations that (geometric) magnitudes comply with. Thus, many reasons advanced to support the former claim can, quite easily, be reversed so as to produce reasons to support the latter. This is, in any case, what happens with Barrow's argument that we are discussing. Barrow himself implicitly acknowledges this, when he remarks that the equality (4) follows from the quadrature of the parabola, provided that "the Agreement of Arithmetic with Geometry [...][is] supposed." His argument is, essentially, the same as that of Wallis: whereas Barrow relies on it for supporting his thesis of a subsumption of the domain of arithmetic under that of geometry, Wallis relies on it for founding an arithmetical method of quadrature. But for this argument to hold, the agreement of arithmetic with geometry has to be supposed, and there is no way to decide, on the basis of this argument, whether such an agreement has to be understood as a reduction of the former to the latter, or of the latter to the former, or also as nothing but a local fruitful cooperation.

Thus, all that Barrow could have pretended to show with his example, in the perspective of the internal evolution of mathematics, is that, in the case under examination, the geometric understanding of the relevant relations—which is a particular form of functional understanding—is able to make manifest the equivalence of the equalities (4) and (2). This is because this understanding allows us to grasp, so to speak, the hidden linearity that is responsible for this equivalence.

Still, Barrow's point is open to another interpretation: not a mathematical one, but rather an outright philosophical one. When Wallis' deduction of the quadrature of the parabola based on the equality (4) is compared with Barrow's inverse deduction of the equality (2) based on this same quadrature, in the light of the conceptual tools of the Aristotelian philosophy of mathematics—rather than of the conceptual tools of mathematical practice—an asymmetry appears. The sort of infinity involved in the equality (2) is actual infinity by addition; the sort of infinity involved in the quadrature of the parabola is potential infinity by division. In the first case, infinity is an attribute of an object—the infinite sum—that appears as being constituted by an infinity of parts. In the second case, infinity is an attribute of a sequence of possible acts which take as their substratum a continuous object given beforehand. Whereas, in the first case, the arithmetical theorem to be proved on a purely arithmetical basis is appropriately expressed by the very equality (2), in the second case, this same theorem is more appropriately expressed by an equality like

$$\frac{\sum_{i=0}^{n} \sqrt{i\varepsilon}}{(n+1)} = \frac{2}{3}, \tag{5}$$

where $\varepsilon$ is an infinitely small portion of $\mathsf{OP}$ which is supposed to be equal to $\frac{1}{n}$ of it (so that $n$ becomes infinite).

I suggest that the only point that Barrow's example can support is that his own deduction of the equality (2), based of the quadrature of the parabola is preferable over Wallis inverse deduction since the latter form of infinity is a legitimate one, whereas the former is not, or, better, since the former form becomes legitimate only if it is understood as an expression of the latter: if a model for it is provided, where the infinity appears to be a potential infinity by division. This is a classical thesis, an Aristotelian thesis, in fact. Moreover, it is a thesis that—sometimes asserted, sometimes denied—appears at different historical junctures. But this is a philosophical thesis: a thesis that has often gone together with an assessment of some mathematical theories, or that has even suggested or promoted the establishment of some of them, but that has never provided the relevant conceptual tools to help mathematics to progress[31].

## 3 Motion and geometric series

I come now to the second example[32]. Here it is:

> *The Sum of a Series of Numbers infinitely decreasing in a triple Ratio [better: "The Series or Sum of Numbers infinitely decreasing continually from Unity (included) in a triple ratio"] (i.e. as* 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, *&c.) till the last or least Term be nothing is to Unity as* 3 *to* 2*; or the same Sum excluding Unity is equal to* $\frac{1}{2}$ [better: "the half of the Unity"]. This Theorem (which for Easiness and Perspicuity sake, I have proposed in a determinate, *viz.* a triple Ratio) although it may be demonstrated Universally, yet I think, by making Use of continued Quantity, it will be shewn much more clearly and expeditiously, at least more handsomely and elegantly. [...] This Theorem universally is as follows. *The Sum of any Series of Numbers [better: "The Series of Numbers"] continually decreasing from Unity infinitely, or to Nothing, in any Proportion, will be to Unity, as the Antecedent, or greater Term of the Proportion, is to the Excess of the Antecedent above the Consequent. Or the*

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[31] In the particular case under examination, this is confirmed by the fact that, in the absence of a functional notion of limit and/or of any version of the calculus, the difference between the equalities (2) and (5) is perfectly ineffective.

[32] Cf. [7], pp. 31-33. The brackets include some suggestions for slight corrections of Kirkby's translation.

> *same Sum, excluding Unity, will be to Unity, as the Consequent, or lesser Term of the Proportion, is to the Excess of the Antecedent above the Consequent.*[...] So easily are Arithmetical conclusions, (otherwise sufficiently intricate and difficult to be investigated) drawn from the Consideration of Geometry. Thus doth the one Science require the help of the other, and after a friendly sort conspire together.

Barrow's claim about the virtue of a geometric proof (or, as he said, of a proof "making Use of continued Quantity", that is, magnitudes) is, in this case, much weaker than in the case of the first example: far from maintaining that the arithmetical theorem under consideration cannot be proved in purely arithmetical terms, he argued that geometry makes it possible to prove such a theorem "much more clearly and expeditiously, at least more handsomely and elegantly", and that any other different proof (that is, any arithmetical one) is "sufficiently intricate and difficult". Such a cautious attitude was imposed by the fact that Barrow was probably aware that, at the time of his lectures, the theorem had been already proved arithmetically at least once, namely by Viète, in a treatise first published in 1593, and reprinted in 1646 within Van Schooten's edition of Viète's *Opera Mathematica*[33]. But, judged from a strictly mathematical point of view, Barrow's attitude is not cautious enough, however, since Viète's proof—or proofs, since, as we shall see later, one could recognize two distinct proofs of the same theorem in Viète's text—are far from being "intricate and difficult", and could be hardy considered as less clear, compact and elegant than Barrow's one. Moreover, the same theorem can also be proved—and was actually proved, after 1664, by many mathematicians—in purely arithmetical, or at least in non-geometric terms, according to quite simple arguments, different from Viète's, that Barrow probably became aware of, or, in any case, would have certainly easily understood. Thus, also in this case, if we want to assign to Barrow's example some argumentative force, we need to consider it from a different point of view.

### 3.1 Barrow's proof

Let us begin with the geometric proof he suggested. This is, in fact, a double one, since it concerns successively the particular case where the base and the ratio of the geometric series are 1 and $\frac{1}{3}$, respectively, and the general case.

In the particular case, the theorem can be stated as follows, in modern notation:

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots=\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^{i}=\frac{3}{2}, \quad \text{that is,} \quad \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots=\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i}=\frac{1}{2}. \tag{6}$$

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[33] Cf. [38], ch. XVII, and [39], pp. 397-398. The conjecture that Barrow knew of Viète's proof is supported by the fact that Van Schooten's edition was part of Barrow's library at the time of his death, in 1677, and was probably there also in 1664: cf. [14] , pp. 368 and 336, respectively.

In the general case, this becomes, of course:

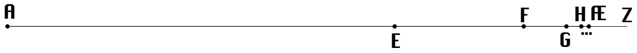
$$1 + x + x^2 + x^3 + \ldots = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \ , \quad \text{that is,} \quad x + x^2 + x^3 + \ldots = \sum_{i=0}^{\infty} x^i = \frac{x}{1-x}, \quad (7)$$

where one should implicitly suppose that $x < 1$. This can also be written in an even more general form, at least apparently. It is enough to replace the unity with any number $\alpha$, as the base or first term of the series, and to suppose that the second term is any other number $\beta$, so that the ratio—or "proportion", as Barrow said—is $\frac{\beta}{\alpha}$. One gets, then:

$$\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} + \ldots = \sum_{i=0}^{\infty} \frac{\beta^i}{\alpha^{i-1}} = \frac{\alpha^2}{\alpha - \beta}, \quad (8)$$

where one should implicitly suppose that $\beta < \alpha$.

Barrow's proof relies on the consideration of a geometrical model associated with the Achilles paradox, that I shall call, for short, "Achilles paradox model". Here is this proof[34]:



> Suppose the movable point A to be carried through the Right Line AZ, with an uniform Motion: Also imagine the point E to be moved uniformly through the same, but with a Velocity subtriple the Velocity wherewith the Point A is carried; then, in whatsoever time the Point A runs through the Right Line AE in the same time will the Point E run through a third part of AE, suppose EF. Also in whatsoever time the former Point runs through the Right Line EF, in the same time will the latter run through the third Part of EF, suppose FG, and so on, *ad infinitum*, till the Point A overtake E in Æ. Now if we suppose the Line AE to be the Unity, then will EF be $\frac{1}{3}$; FG $\frac{1}{9}$, and GH $\frac{1}{27}$, &c. *ad infinitum*, according to the Hypothesis; but it appears that the line A[[Æ]] run through by A is triple the Line EÆ which is run through by E in the same time, because A is supposed to move with three times the Velocity of E. Therefore EÆ is to AE[[, hoc est series decrescens, exclusâ unitate, ad unitatem, ut 1 ad 2: Et AÆ se habet ad AE]] (*i. e.* the decreasing

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[34] Cf. [7], pp. 32-33. The simple brackets include some suggestions for slight corrections of Kirkby's translation. The double brackets include some other corrections relative to three typographical errors in the English translation of Kirkby (only one of which also appears in the original Latin text [cf. [9], pp. 36-37): in the first case the English text has "E", instead of "Æ" that correctly occurs in the original text; in the second case the part in double brackets, that occurs in the original text, is omitted; in the third case, both the English and the original texts have "AE : EÆ :: $R - S : S$", instead of "AÆ : EÆ :: $R : S$", as would be correct (this typographical error is corrected in [8] , p. 49).

> Series including Unity is to Unity) as 3 to 2. Q. E. D. [. . . ] For the foregoing Premisses being supposed, let the Ratio be given between $R$ and $S$, and let AE represent [better: "be"] Unity, as above; then because AÆ (*i. e.* AE+EÆ) : EÆ : : $R$ : $S$. therefore (by dividing the Proportion) AE : EÆ :: $R - S$ : $S$. *i. e.* Unity is to the Sum of the proposed Series [better: "to the proposed Series"], as the Excess of the Terms of the Proportion is to the lesser Term: And again, because [[AÆ : EÆ :: $R$ : $S$]]. therefore AÆ : AE :: $R$ : $R - S$, by Conversion, *i. e.* the proposed Sum [better: "the proposed Series"] including Unity, is to Unity, as the greater Term of the Proportion is to its Excess above the lesser.

Let us suppose two "movable points" to be moved with uniform rectilinear motion on a straight line AZ, starting from two fixed points A and E. Let us also suppose that the velocity of the first of these points is the velocity of the second as 3 to 1, in the particular case, or as $R$ to $S$ ($S < R$), in the general case. Let us finally suppose that the first point reaches the second at Æ. It follows that

$$\text{AÆ} : \text{EÆ} = 3 : 1 \quad \text{or} \quad \text{AÆ} : \text{EÆ} = R : S, \tag{9}$$

and, as AE = AÆ−EÆ,

$$\text{AÆ} : \text{AE} = 3 : 2 \qquad \text{and} \qquad \text{EÆ} : \text{AE} = 1 : 2,$$

or

$$\text{AÆ} : \text{AE} = R : R - S \qquad \text{and} \qquad \text{EÆ} : \text{AE} = S : R - S.$$

But, to come to Æ, the two points pass through the points E, F, G, H,. . . and F, G, H,. . . , respectively, where they are at the same time. Thus

$$\begin{gathered} \text{AE} : \text{EF} = \text{EF} : \text{FG} = \text{FG} : \text{GH} = \ldots = 3 : 1 \\ \text{or} \\ \text{AE} : \text{EF} = \text{EF} : \text{FG} = \text{FG} : \text{GH} = \ldots = R : S, \end{gathered} \tag{10}$$

and hence:

$$\begin{aligned} \text{AÆ} \quad &= \text{AE} + \text{EF} + \text{FG} + \text{GH} + \ldots \quad &&= \text{AE} + \frac{1}{3}\text{AE} + \frac{1}{3}\text{EF} + \frac{1}{3}\text{FG} + \ldots \\ & &&= \text{AE} + \frac{1}{3}\text{AE} + \frac{1}{3}\frac{1}{3}\text{AE} + \frac{1}{3}\frac{1}{3}\frac{1}{3}\text{AE} + \ldots \end{aligned} \tag{11}$$

or

$$\begin{aligned} \text{AÆ} \quad &= \text{AE} + \text{EF} + \text{FG} + \text{GH} + \ldots \quad &&= \text{AE} + \frac{S}{R}\text{AE} + \frac{S}{R}\text{EF} + \frac{S}{R}\text{FG} + \ldots \\ & &&= \text{AE} + \frac{S}{R}\text{AE} + \frac{S}{R}\frac{S}{R}\text{AE} + \frac{S}{R}\frac{S}{R}\frac{S}{R}\text{AE} + \ldots \end{aligned} \; . \tag{12}$$

It is thus enough to suppose that AE is, or represents unity, to conclude that

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots=\frac{3}{2} \qquad \text{and} \qquad \frac{1}{3}+\frac{1}{9}+\frac{1}{27}=\frac{1}{2}, \tag{13}$$

or

$$1+\frac{S}{R}+\left(\frac{S}{R}\right)^2+\left(\frac{S}{R}\right)^3+\ldots=\frac{R}{R-S} \qquad \text{and} \qquad \frac{S}{R}+\left(\frac{S}{R}\right)^2+\left(\frac{S}{R}\right)^3+\ldots=\frac{S}{R-S}, \tag{14}$$

according to (6) and (7), respectively.

If one supposes, instead, that AÆ, AE, and EF are or represent any triple of quantities $Z$, $\alpha$, and $\beta$ (such that $\beta<\alpha<Z$), one gets:

$$Z : Z-\alpha = \alpha : \beta,$$

instead of (9), and hence, because of (10):

$$Z=\frac{\alpha^2}{\alpha-\beta}=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}+\ldots, \tag{15}$$

according to (8). This is just what Barrow remarks in a short, purely formalistic, footnote inserted after the claim that the theorem "may be demonstrated Universally[35]". The footnote specifies how it might be done. Here is this footnote:

> Thus:
>
> $$Z-o : Z-\alpha :: \alpha : \beta.$$
>
> $$Z\beta = Z\alpha - \alpha\alpha.$$
>
> $$\alpha\alpha = Z\alpha - Z\beta.$$
>
> $$\frac{\alpha\alpha}{\alpha-\beta}=Z.$$

### 3.2 Is Barrow's proof essentially geometric?

Barrow's proof is quite easy and, of course, it is right. But, as already said, an equally easy arithmetical proof for the same theorem was quite certainly known by Barrow himself. Before considering the latter in detail, one should answer a preliminary question concerning the former: is this a genuine geometric proof? Of course, what is relevant is not whether it is geometric or kinematic (since we can, for our purpose, admit Barrow's view, according to which kinematics—or at least the small fragment of kinematics that is used in this proof—and geometry do not essentially differ). The point is rather whether its geometric features are essential for the success of the argument, and whether these depend or not on more fundamental arithmetic conditions.

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[35] Cf. [7], pp. 32.

The Achilles paradox model plays a double role in Barrow's proof. Firstly, it provides a model for the geometric series. This is not only a geometric model. It is also, so to speak, a phenomenal one: it includes both an infinite partition of a finite segment, and an interpretation of this segment and its partition as geometric representations of a physical phenomenon. The model is thus strong enough to ensure that, in so far as the ratio of a geometric series is smaller than 1 (or, more generally of the ratio that any quantity has with itself), this series has a (finite) sum, and to represent this sum under the form of an already given segment: the segment AÆ. The proof of the theorem is thus reduced to the solution of a problem: to determine the quantitative relations that this segment has with its parts under the relevant partition. Secondly, the Achilles paradox model suggests the way to solve this problem, since it goes together with the establishment of the proportions (9). These proportions being given, and the interpretation of the geometric series provided by this model being admitted, the solution of the problem is indeed a trivial exercise.

Now, let us suppose that the existence of a (finite) sum for a geometric series is admitted on different and independent bases. Of course, this is not the same as supposing that this sum is known. But this sum is not known under the interpretation provided by the Achilles paradox model either. It is represented by a segment, but this segment is, in fact, unknown. To trace such a segment and take it as if it were known, as Barrow suggests, is thus part of an analytic practice. A similar practice can also be applied to the (finite) sum of a geometric series that, according to our supposition, has been admitted to exist: let $U$ be this sum. We shall have:

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots=U \qquad \text{or} \qquad 1+\frac{S}{R}+\left(\frac{S}{R}\right)^2+\left(\frac{S}{R}\right)^3+\ldots=U. \tag{16}$$

As a matter of fact, this is a sufficient premise to prove the theorem in a purely arithmetic way, since from it, it follows that:

$$\begin{aligned} U-1 &= \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots \\ &= \frac{1}{3}\left(1+\frac{1}{3}+\frac{1}{9}+\ldots\right) \\ &= \frac{1}{3}U \end{aligned} \qquad \text{or} \qquad \begin{aligned} U-1 &= \frac{S}{R}+\left(\frac{S}{R}\right)^2+\left(\frac{S}{R}\right)^3+\ldots \\ &= \frac{S}{R}\left(1+\left(\frac{S}{R}\right)+\left(\frac{S}{R}\right)^2+\ldots\right) \\ &= \frac{S}{R}U \end{aligned}$$

and thus

$$\frac{2}{3}U=1 \qquad \text{or} \qquad \frac{R-S}{R}U=1,$$

according to (13) and (14), respectively.

The equalities $U - 1 = \frac{1}{3}U$ and $U - 1 = \frac{S}{R}U$ are equivalent, of course, to the proportions

$$U : U - 1 = 3 : 1 \qquad \text{and} \qquad U = U - 1 = R : S,$$

which are, in turn, equivalent to the proportions (9). On the other hand, the equalities (11) and (12) add no information which supplements that in the equalities (16): they simply correspond to these last equalities under the interpretation provided by the Achilles paradox model. It follows that, once the existence of a (finite) sum for the geometric series is admitted without relying on the Achilles paradox model—that is, when other and independent reasons for supporting this admission are provided—the theorem can be proved, in purely arithmetical terms, according to a line of argumentation that parallels Barrow's geometric argument. The second role played in this proof by the Achilles paradox model is thus far from essential: the proof can be repeated, *mutatis mutandis*, without relying on such a setting relative to this role, provided that the existence of a (finite) sum for the geometric series is admitted on independent bases.

On the other hand, one could wonder whether the admission of the existence of a sum for a geometric series and its expression by a term like "$U$" are essential for conducting the previous proof. It is easy to understand that they are not. It is rather enough that a geometric series itself is handled as a (finite) quantity. If this is admitted, the proof can be rephrased as follows:

$$\left[\begin{array}{ll} 1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots & = 1 + \left[\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots\right] \\ & = 1 + \dfrac{1}{3}\left[1 + \dfrac{1}{3} + \dfrac{1}{9} + \ldots\right] \end{array}\right] \Rightarrow \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right] = 1$$

or

$$\left[\begin{array}{ll} 1 + \dfrac{S}{R} + \left(\dfrac{S}{R}\right)^2 + \left(\dfrac{S}{R}\right)^3 + \ldots & = 1 + \left[\dfrac{S}{R} + \left(\dfrac{S}{R}\right)^2 + \left(\dfrac{S}{R}\right)^3 + \ldots\right] \\ & = 1 + \dfrac{S}{R}\left[1 + \dfrac{S}{R} + \left(\dfrac{S}{R}\right)^2 + \ldots\right] \end{array}\right] \Rightarrow$$
$$\Rightarrow \frac{R - S}{R}\left[1 + \frac{S}{R} + \left(\frac{S}{R}\right)^2 + \left(\frac{S}{R}\right)^3 + \ldots\right] = 1.$$

Barrow's proof is thus, as a matter of fact, a geometric version of a purely arithmetical proof based on the supposition that a geometric series has a (finite) sum or can, at least, be handled as a (finite) quantity: the result of an interpretation of such a proof according to a model as vivid as the Achilles paradox one. This interpretation is certainly elegant and heuristically evocative, but it has no relevant mathematical function in the proof of the theorem other than that of warranting the existence of a (finite) sum of a geometric series or, at least, the possibility of handling this series as a (finite) quantity.

Still, it openly suggests a generalization of this theorem to any sort of quantity, namely its extension to magnitudes. Under this extension, this theorem ceases to be an arithmetical theorem and cannot, of course, have a purely arithmetical proof. The equivalence between Barrow's proof, relying on the Achilles paradox model, and its underlying arithmetical proof has, thus, an important mathematical function: that of showing which operational properties a domain of magnitudes has to satisfy for the sum of a geometric series to be definable and provably determinable on such a domain. Both the notation that Barrow uses in the general version of his proof, and the short, purely formalistic text of his footnote that provides a proof of the equality (15) suggest that Barrow was willing to implicitly point out such a possible extension. The symbols "$R$", "$S$", "$Z$", "$\alpha$", and "$\beta$" in the context of these proofs, have, indeed, a quite natural interpretation as symbols designating any sort of quantity, namely numbers or continuous magnitudes: note that, whereas Barrow writes the fraction "$\frac{1}{3}$", giving the ratio of the geometric series in the particular case, he does not write the corresponding fractions "$\frac{S}{R}$", and "$\frac{\alpha}{\beta}$ ", and, when he writes the fraction "$\frac{\alpha\alpha}{\alpha-\beta}$", he is, quite probably, as we shall see, quoting Viète implicitly.

But, if the Achilles paradox model is not understood as a geometric model of a configuration of arithmetical objects, that is, numbers, but rather as a kinematic interpretation of the geometric configuration attached to a geometric series defined on the domain of segments—the domain where any relation between magnitudes was usually represented in the modern age—, then it ceases to be a basis for a geometric reformulation of an arithmetical proof of an arithmetical theorem. It rather becomes a basis for interpreting a general theorem concerning any sort of quantity, and specially magnitudes, on the domain of segments. In 1664, two possible contexts for stating a similar theorem, and proving it, were available[36]: Viète's common algebra, where multiplication and division were axiomatically defined on quantities of any sort; and Descartes' geometric algebra, where multiplication and division were explicitly defined on magnitudes by relying on appropriate proportions. In both cases, far from being reduced to geometry, arithmetic provided an explicit model for a formalism to be used to make geometry in a new way. Thus, although it suggests the possibility of extending the theorem that Barrow explicitly refers only to numbers also to magnitudes, Barrow's geometric proof of this theorem is quite far from supporting the thesis of a subsumption of the domain of arithmetic under that of geometry.

From these considerations it follows that such a proof can be understood as essentially geometric only in so far as the appeal to a geometric model—like the Achilles paradox one—is conceived as the only way to warrant the existence of a (finite) sum for a geometric series, or, at least, to license the manipulation of such a series as a (finite) quantity. Were one ready to admit the existence of such a sum or the possibility of a similar manipulation of a geometric series, this proof should not be understood as essentially geometric. Moreover, its reformulation as a purely arithmetical proof is enough to show that it is not clearer, more compact and elegant than any arithmetical proof.

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[36] Cf. [28], Introduction, and p. 3 above.

### 3.3 Viète's proof

This last claim is also confirmed by the consideration of Viète's proofs[37].

Chapter XVII of Viète's *Variorum de rebus mathematicis responsorum, liber VIII*[38] has a quite explicit title: "Progressio Geometrica". It is a very short chapter, since, as Viète said, the whole theory of geometric progressions can be summed up in only one theorem, together with four simple corollaries that, by alluding to Euclid's *Data*[39], Viète calls "*δεδομένα*"[40]:

> Progressionis Geometricæ doctrina uno ferè absoluitor Theoremate, diductis videlicet ex eo quatuor *δεδομένοις*.

Here is this theorem, as it appears in Van Schooten's edition of Viète's *Opera Mathematica*[41], where Barrow probably read it[42]:

> Theorema. Si fuerint magnitudines continue proportionales: erit ut terminus rationis major ad terminum rationis minorem, ut differentia compositæ ex omnibus & minimæ ad differentiam compositæ ex omnibus & maximæ.
>
> Sint magnitudines continue proportionales, quarum maxima sit $D$, minima $X$, & composita ex omnibus $F$, & sit ratio majoris ad minorem sicut $D$ ad $B$. Dico esse ut $D$ ad $B$, ita $F$ minus $X$ ad $F$ minus $D$.

In Viète's original treatise, the first part of the theorem is different[43]:

> Si fuerint magnitudines continuè proportionales, Erit ut terminus rationis maior ad terminum rationis minorem, ita composita ex omnibus ad differentiam compositæ ex omnibus & maximæ.

The difference is all the more surprising as it does not repeat itself in the second part of the theorem, that, in the original text of Viète, is thus inconsistent with the first part.

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[37] My attention was first drawn to this proof by Jean Dhombres who—in his [11], pp. 149-154—has suggested it provides an example of a "suspended analysis".

[38] Cf. [38], pp. 28*v*-29*r*, and [39], pp. 397-398.

[39] Cf. [11], p. 151.

[40] Cf. [38], p. 28*v*, and [39], p. 397.

[41] Cf. [39], p. 397.

[42] Cf. [39], p. 397. Cf. also the previous footnote (33).

[43] Cf. [38], p. 28*v*.

In Viète's notation, $D$ and $X$ are, respectively, the first and the last term of the progression, which are also the greater and the smaller, since the progression is taken to be decreasing; $F$ is its sum, and its ratio is that of $D$ to $B$, so that $B$ is the second term. Hence, in Van Schooten's edition, the first and second part of the theorem agree in stating that

$$D : B = F - X : F - D. \tag{17}$$

It seems thus, under this version of Viète's statement, that the progression is finite, and is, thus, not yet a series, in our sense. The existence of its sum $F$ would then be unproblematic, and Viète's theorem would amount to an obvious consequence of the conjunction of the propositions V.12 and VII.12 of the *Elements*, since it seems obvious that Viète included (positive integer) numbers among what he called "magnitudes", in the statement of the theorem.

Here are Euclid's propositions, in Heath's translation[44]:

> [V.12] If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.
>
> [VII.12] If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Using indexes, and supposing that $\alpha_i$ and $\beta_i$ $(i = 1, 2, \dots, n)$ are $2n$ quantities such that

$$\alpha_1 : \beta_1 = \alpha_2 : \beta_2 = \alpha_3 : \beta_3 = \ldots = \alpha_{n-1} : \beta_{n-1} = \alpha_n : \beta_n, \tag{18}$$

these propositions assert that

$$(\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} + \alpha_n) : (\beta_1 + \beta_2 + \beta_3 + \ldots + \beta_{n-1} + \beta_n) = \alpha_1 : \beta_1, \tag{19}$$

provided that $\alpha_1 : \beta_1 = \alpha_\lambda : \beta_\lambda$, for any $\lambda$ such that $1 \leq \lambda \leq n$. Euclid's proofs rely on his definitions of proportion between magnitudes and (positive integer) numbers—the very well known definitions V.5 and VII.20—and to appropriate linearity properties of magnitudes and (positive integer) numbers, established in propositions V.1, VII.5 and VII.6, that is:

$$\text{if } a = ha',\ b = hb',\ \ldots z = hz',\ \text{then } a + b + \ldots + z = h\left(a' + b' + \ldots + z'\right)$$

and

$$\text{if } h\alpha_1 = k\beta_1 \text{ and } h\alpha_2 = k\beta_2,\ \text{then } h\left(\alpha_1 + \alpha_2\right) = k\left(\beta_1 + \beta_2\right),$$

where $a$, $b$,…, $z$, and $a'$, $b'$,…, $z'$ are whatever magnitudes and $h$ and $k$, are whatever (positive integer) numbers. In both cases the proof is immediate. Supposing that the fundamental properties of proportions are admitted, the same propositions can, on the other hand, be proved by reiterating the following argument as many times as necessary[45]:

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[44] Cf. [12], vol. 2, pp. 159 and 312.

$$\begin{array}{lll} \text{if} \quad \alpha_1 : \beta_1 = \alpha_2 : \beta_2 & \text{then} & \alpha_1 : \alpha_2 = \beta_1 : \beta_2, \\ & \text{and hence} & (\alpha_1 + \alpha_2) : \alpha_1 = (\beta_1 + \beta_2) : \beta_2, \\ & \text{that is,} & (\alpha_1 + \alpha_2) : (\beta_1 + \beta_2) = \alpha_1 : \beta_2. \end{array}$$

Whatever proof one adopts, the implication 'if the multiple proportion (18) holds, then the (simple) proportion (19) holds' is quite unproblematic in the tradition of Euclid's mathematics. To draw, from it, the proportion (17), is then enough to suppose that $\alpha_{i+1} = \beta_i$ $(i = 1, 2, \ldots, n-1)$, then replace $\alpha_1$ with $D$, $\beta_1 = \alpha_2$ with $B$, and $\beta_n$ with $X$. Under these replacements, from the continuous proportion

$$D : B = B : \beta_2 = \beta_2 : \beta_3 = \ldots = \beta_{n-2} : \beta_{n-1} = \beta_{n-1} : X,$$

one gets, indeed, according to this implication:

$$(D + B + \beta_2 + \ldots + \beta_{n-2} + \beta_{n-1}) : (B + \beta_2 + \ldots + \beta_{n-2} + \beta_{n-1} + \beta_n) = D : B,$$

that is just the proportion (17), provided that

$$D + B + \beta_2 + \ldots + \beta_{n-2} + \beta_{n-1} + \beta_n = F.$$

The second part of Viète's theorem—or this whole theorem, in Van Schooten's version—is thus a triviality in the tradition of Euclid's mathematics. Viète's new algebraic formalism could have allowed to add very little to such a triviality: just an explicit way to denote the terms of the progression included between $D$ and $X$, so as to write the geometric progression under the form

$$D + B + \frac{B^2}{D} + \frac{B^3}{D^2} + \ldots + \frac{B^n}{D^{n-1}},$$

where $\frac{B^n}{D^{n-1}} = X$, and thus to express, openly, the fact that these terms are in a continuous proportion to each other[46].

It is plausible to guess that Van Schooten's version of the theorem depends on nothing but a correction of a banal material mistake made by Viète himself because of his eagerness

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[45] But remark that this argument appeals to the proposition V.16, which asserts that if $\alpha_1 : \beta_1 = \alpha_2 : \beta_2$, then $\alpha_1 : \alpha_2 = \beta_1 : \beta_2$, and is proved, in turn, by Euclid, by appealing to V.12.

[46] But remark that, once the progression is so written, there is room to prove Viète's theorem in an alternative way, by observing that

$$D + B + \frac{B^2}{D} + \frac{B^3}{D^2} + \ldots + \frac{B^n}{D^{n-1}} = \frac{D^n + D^{n-1}B + D^{n-2}B^2 + \ldots B^n}{D^{n-1}},$$

and

$$D^{n+1} - B^{n+1} = (D - B)\left(D^n + D^{n-1}B + \ldots + DB^{n-1} + B^n\right),$$

to state his own relevant addition to the previous Euclidean result. If it were so, there would be no reason to be surprised that Viète offered no proof of his theorem, and that he used his algebraic formalism only to draw from the proportion (17) the announced corollaries, that, according to such a formalism, immediately follow from this theorem, whatever the nature of the quantities $D$, $B$, and $X$. Here are these corollaries[47]:

$$\begin{aligned} &i) \quad F = \frac{D^2 - BX}{D - B}; \\ &ii) \quad X = \frac{BF + D^2 - DF}{B}; \\ &iii) \quad B = \frac{DF - D^2}{F - X}; \\ &iv) \quad DF - D^2 = BF - BX. \end{aligned} \tag{20}$$

Of course, it is the first corollary that mainly interests us. If one adopts Viète's formalism to express the terms of the geometric proportion, and writes "$\frac{B^n}{D^{n-1}}$" in place of "$X$", one gets from it the following equality:

$$D + B + \frac{B^2}{D} + \frac{B^3}{D^2} + \ldots + \frac{B^n}{D^{n-1}} = \frac{D^2 - \frac{B^{n+1}}{D^{n-1}}}{D - B}.$$

Though there is no doubt that Viète understood his progression as decreasing (supposing that $B < D$), this equality holds for any geometric progression whose ratio differs from unity (that is, also if $D < B$). Still, whereas in the second case, as the term $X = \frac{B^{n+1}}{D^{n-1}}$ progressively increases, the sum $F = \frac{D^2 - \frac{B^{n+1}}{D^{n-1}}}{D-B}$ becomes infinite with $n$, in the first case, since the term $X = \frac{B^{n+1}}{D^{n-1}}$ progressively decreases, when $n$ becomes infinite such a term vanishes and the sum $F = \frac{D^2 - \frac{B^{n+1}}{D^{n-1}}}{D-B}$ is then transformed in the finite ratio $\frac{D^2}{D-B}$. This is just what Viète remarked in the second part of his chapter, and possibly had in his mind when he wrote, unadvisedly, the first part of his theorem as it appears in the original edition of his treatise.

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so that:

$$D + B + \frac{B^2}{D} + \frac{B^3}{D^2} + \ldots + \frac{B^n}{D^{n-1}} = \frac{D^{n+1} - B^{n+1}}{(D - B)\,D^{n-1}} = \frac{D^2 - B\frac{B^n}{D^{n-1}}}{D - B}.$$

It is perhaps of some interest to remark that the second of the previous equalities is the fundamental equality of Sluse's method of tangents, a method that Sluse was probably elaborating at the beginning of the sixties and that he exposed, some years later in two letters that were published in the *Philosophical Transactions*: cf. [35] and [36].

[47] Cf.[38], p. 29*r*, and [39], p. 397. Of course, Viète writes "∼ *quadratum*" or "∼ *quadrato*" (abridged in "∼ *quad.*" in [39]) where we write "$\sim^2$" and "∼ *in* ∼" where we write "∼∼". This apart, his notation is the same than ours. In [39] there is a typographical error in the first corollary, since "*B quad.*" appears in place of "*D quad.*"

Here is what he wrote[48]:

> An verò cum magnitudines sunt continuè proportionales in infinitum, abibit $X$ in nihilum. Et evanescere asserent Mechanici, cum minima quantitas subsit tantum intellectu.
>
> *Itaque erit secundum eos. Ut differentia terminorum rationis ad terminum rationis majorem, ita maxima ad compositam ex omibus.*

There is no doubt that Viète was arguing that, when the geometric progression proceeds to infinity, the proportion (17) reduces to

$$D - B : D = D : F_\infty, \tag{21}$$

where $F_\infty$ is just the sum of the geometric progression continued to infinity, that is, the geometric series[49], and there is also no doubt that he was admitting that a possible argument for this claim is that, when the progression proceeds to infinity, $X$ vanishes. If one puts, indeed, $X = 0$ and $F = F_\infty$, in the proportion (17) and in the equality (20.$i$), one gets, respectively:

$$D : B = F_\infty : F_\infty - D \quad \text{and} \quad F_\infty = \frac{D^2}{D-B},$$

which are both equivalent to the proportion (21). It is sufficient to replace $F_\infty$, $D$, and $B$ with $Z$, $\alpha$, and $\beta$, to understand that Viète has thus provided a proof of the equality (15), and proved Barrow's theorem, in the general case where the terms of the geometric series are any sort of quantity.

But it is also clear that, for Viète, this argument did not have the same nature as an "analytic" one. Rather it was a "mechanical" argument, that is—if I understand well—an argument based, merely, on successive approximate estimates. Thus, Viète considered it necessary to offer another argument for the same conclusion.

Here is what he wrote, firstly[50]:

> *Cum alioquin esset, Ut differentia terminorum rationis ad terminum rationis minorem, ita minima ad crementum.*
>
> *Et ut differentia terminorum rationis ad terminorum rationis maiorem, ita maxima ad compositam ex omnibus plus cremento.*

According to Dhombres[51], the first of these two proportions provides a definition for the "increment" relative to the last term of the progression, that is, $X$. The second is, instead, a consequence of the equality (20.$i$), together with this last definition. If the increment $\Delta_X$ (or $\Delta_n$, if one prefers, provided that $X = \frac{B^{n+1}}{D^{n-1}}$) is so defined that

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[48] Cf. [38], p. 29$r$, and [39], p. 398.

[49] The symbol "$F_\infty$" is of course not used by Viète. I introduce it for simplicity.

[50] Cf. [38], p. 29$r$, and [39], p. 398.

[51] Cf. [11], p. 152.

$$D - B : B = X : \Delta_X, \tag{22}$$

from the equality (20.$i$) it follows that

$$F = \frac{D^2}{D-B} - \frac{BX}{D-B} = \frac{D^2}{D-B} - \Delta_X \tag{23}$$

that is:

$$D - B : D = D : F + \Delta_X,$$

according to the second proportion. The proportion (21) follows from this last proportion, provided that $\Delta_X$ vanishes with $X$. This last condition is certainly necessary for the proportion (22) to hold, in this case. But to rely on it in order to prove the proportion (21) is the same as relying on the supposition that the proportion (22) continues to hold when $X$ vanishes, and thus, implicitly, to the supposition that $X$ vanishes when the progression proceeds to infinity. Hence, if this were Viète's argument, it would be as little "analytic" as the previous one.

But this is not Viète's argument. Here is what he wrote after the passage quoted before[52]:

> *Sint magnitudines proportionales in continuâ ratione subquadruplâ εἰς ἄπειρον, & sit maxima omnium* 3. *Composita ex omnibus fiet* 4. *Neque enim magnitudinibus illis in continuâ ratione subquadruplâ existentibus quarum maxima est* 3 *tantula potest addi, quin composita sit major* 4. *Eoque pertinet quadratio Paraboles Archimedæa.*

This could appear to be an argument by authority. Still, it is not so. Certainly Viète was referring here to a particular result obtained by Archimedes in the *Quadrature of the Parabola*, namely to the propositions 23 and 24 of this treatise[53]. But he was also, and overall, implicitly suggesting a possible application—or better, a generalization—of Archimedes' proof by exhaustion, to the general case of any (decreasing) geometric series.

In the proposition 23 of his treatise, Archimedes had proved a particular case of the equality (20.$i$), namely that

$$A + B + C + \ldots + Y + Z + \frac{1}{3}Z = \frac{4}{3}A, \tag{24}$$

where $A$, $B$, …, $Z$ are any number of magnitudes that are to each other as 4 to 1, that is, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, …, $Z = \frac{1}{4}Y$. Archimedes' proof is quite different from the previous one based on the propositions V.12 and VII.12 of the *Elements*. Though finitary, of course, it is rather similar, in spirit, to the arithmetical version of Barrow's proof, which I have reconstructed above. Suppose that $b = \frac{1}{3}B$, $c = \frac{1}{3}C$, ..., $z = \frac{1}{3}Z$, so that

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[52] Cf. [38], p. 29$r$, and [39], p. 398.

[53] For short, I adopt here Heath's transcription of Archimedes' results and arguments: cf. [1], pp. 249-252. For the Greek text and its translation in Latin and French, cf. [3], vol. II, pp. 310-315 and [2], vol. II, pp. 192-195.

$$B + b = \frac{1}{3}A, \quad C + c = \frac{1}{3}B, \quad \dots \quad Z + z = \frac{1}{3}Y;$$

it follows that

$$B + C + \ldots + Y + Z + b + c + \ldots + y + z = \frac{1}{3}(A + B + \ldots + Y)$$

and

$$b + c + \ldots + y = \frac{1}{3}(B + C + \ldots + Y),$$

and thus, by subtraction:

$$B + C + \ldots + Y + Z + z = \frac{1}{3}A,$$

which is equivalent to the equality (24).

This last equality provides a lemma for Archimedes' proof of the proposition 24. There, Archimedes considers a sequence of triangles, $A$, $B$, ..., $Y$, $Z$ so constructed inside a segment of a parabola (the figure delimited by the parabola itself and any one of its chords, parallel to the tangent at its vertex) that $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, ..., $Z = \frac{1}{4}Y$, and such that their sum continuously approaches such a segment from below. He proves, then, by exhaustion, that the segment of parabola is to the first of these triangles as 4 to 3, that is, it is equal to $\frac{4}{3}A$. The proof goes as follows. Suppose that the segment of parabola, let us say $\Pi$[54], is greater than $\frac{4}{3}A$. It would then be possible to take so many triangles $A$, $B$, ..., $Z$ so that the difference $\Pi - \frac{4}{3}A + \frac{1}{3}Z$ between the segment of parabola and their sum becomes smaller than the difference between $\Pi$ and $\frac{4}{3}A$, which is impossible since, whatever $Z$ may be, the inequality

$$\Pi - \frac{4}{3}A + \frac{1}{3}Z > \Pi - \frac{4}{3}A.$$

holds true. Suppose, then, that $\Pi$ is smaller than $\frac{4}{3}A$. It would then be possible to take so many triangles $A$, $B$, ..., $Z$ so that $Z$—and thus, *a fortiori*, $\frac{1}{3}Z$—becomes smaller than the difference between $\frac{4}{3}A$ and $\Pi$, which is impossible since from $\frac{1}{3}Z < \frac{4}{3}A - \Pi$, it follows that

$$A + B + \ldots + Z = \frac{4}{3}A - \frac{1}{3}Z > \Pi,$$

against the supposition that the sum $A + B + \ldots + Z$ approaches $\Pi$ from below.

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[54] The symbol "$\Pi$" does not appear in Archimedes' text. I introduce it for simplicity.

Supposing that in the equality (23), one replaces $D$ with $A$, $B$ with $\frac{1}{4}A$ and $X$ with $Z$, one gets:

$$F = \frac{A^2}{A - \frac{1}{4}A} - \frac{\frac{1}{4}AZ}{A - \frac{1}{4}A} = \frac{4}{3}A - \frac{1}{3}Z = \frac{4}{3}A - \Delta_Z.$$

Archimedes' equality (24) is thus consistent with Viète's equalities (20.$i$) and (23), and the term $\frac{1}{3}Z$ which appears in the first of these equalities is nothing but the particular value of $\Delta_X$ for $\frac{D}{B} = \frac{1}{4}$ and $X = Z$.

It is thus quite natural to understand Viète's mention of Archimedes' result as a sketch of a possible generalization of Archimedes' proof by exhaustion. This would simply work as follows. Suppose that, when the progression proceeds to infinity, its sum $F_\infty$ is greater than $\frac{D^2}{D-B}$. As $\frac{B}{D-B}$ is constant and $\Delta_X = \frac{B}{D-B}X$, it would then be possible to take so many terms $D$, $B$, $\ldots$, $X$ so that the difference $F_\infty - \frac{D^2}{D-B} + \Delta_X$ between the total sum $F_\infty$ and the partial sum $F = \frac{D^2}{D-B} - \Delta_X$ is smaller than the difference between $F_\infty$ and $\frac{D^2}{D-B}$, which is impossible since, whatever $\Delta_X$ may be, the inequality

$$F_\infty - \frac{D^2}{D-B} + \Delta_X > F_\infty - \frac{D^2}{D-B}.$$

holds true. Suppose, then, that $F_\infty$ is smaller than $\frac{D^2}{D-B}$. It would then be possible—again, since $\frac{B}{D-B}$ is constant and $\Delta_X = \frac{B}{D-B}X$—to take so many terms $D$, $B$, $\ldots$, $X$ so that $\Delta_X$ is smaller than the difference between $\frac{D^2}{D-B}$ and $F_\infty$, which is impossible since from $\Delta_X < \frac{D^2}{D-B} - F_\infty$, it follows that

$$D + B + \ldots + X = F = \frac{D^2}{D-B} - \Delta_X > F_\infty.$$

against the supposition that the series $A + B + \ldots + Z$ is decreasing.

Though Archimedes' proof by exhaustion is certainly geometric, the generalization of it that Viète seems to suggest is not, since it simply relies on the relations between the terms of any geometric series and is based on the equality (23) that is proved, both by Viète and Archimedes, without relying on any geometric argument.

Now, the relevant question is: is this second proof advanced by Viète essentially different from the first—mechanical—one? From our point of view it seems to be so, since it seems to be a rudimentary version of a $\varepsilon$-$\delta$ argument. But our point of view could not have been that of Viète: whereas we see in this proof the announcement of something that will come later, Viète could only see a tentative generalization of a particular proof by exhaustion, concerning a particular geometric construction. So it is far from sure that Viète could have seen what, to us, makes the crucial difference between his first and his second argument. And the conclusive phrase of his chapter confirms that he had some doubts about the fact that, unlike the former, the latter was "analytic"[55]:

> Sed vix adsentientur Platonici, cum ipsa omnis Geometria sit intellectu.

Still, this is not a reason to conclude that Barrow could have had some reason to claim that, once restricted to the case where the terms of the geometric series are numbers, Viète's proofs are not purely arithmetic. If one had argued that it is so for this proof is infinitary, one should also have maintained that, for the very same reason, Barrow's theorem itself is not arithmetic. But it is hard to think that Barrow would have advanced a similar argument. The quite schematic proof that he offers for the equality (15), in the footnote we have quoted before is, indeed, nothing but a sketch of the first of Viète's proofs, since the proportion

$$Z - o : Z - \alpha : \ : \alpha : \beta$$

is, manifestly, a rewriting of Viète's proportion (17) with $o$ in place of $X$[56]. On the other hand, it would have been quite hard to argue that, from a mathematical point of view, Viète's proofs are less simple, clear, compact and elegant than Barrow's one. If I have taken so much time to reconstruct them, it is, merely, because I have tried to make manifest their classical—that is, Euclidean and Archimedean—sources. And certainly Barrow would have not considered the classical argument I have mentioned as less virtuous than his own[57].

Then, where could Barrow have seen the advantages of his own proof over Viète's? The only possible answer that I see is that he could have maintained that the Achilles paradox model supplied Viète's implicit generalization of Archimedes' particular proof of the proposition 24 of the *Quadrature of the Parabola* with, so to speak, a concrete framework, similar to that which the segment of parabola and the sequence of triangles constructed inside it supplies Archimedes' proof with. But the need for a similar framework only arises if the sort of infinity involved both in Viète's first proof and in his implicit generalization of Archimedes' proof is seen with suspicion. And one can take this need to be satisfied by the Achilles paradox model only if one considers that the sort of infinity involved in this model is essentially different than the first one.

Thus, we come back to a similar conclusion as that we had reached in section 3.2: the only reason that Barrow could have had to prefer his own proof over Viète's is the same as the only one he could have relied on in order to maintain that his proof is essentially geometric. This reason would have just been the following: only the Achilles paradox model, or some similar geometric models, could license the sort of infinitary argument involved both in the arithmetic proofs expounded in section 3.2 and in Viète's.

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[55] Cf. [38], p. 29*r*, and [39], p. 398. Remember that Viète was agreeing with a traditional belief, quite widespread in his times, according to which the origins of analysis, as an argumentative form, went back to Plato: cf. [37], p. 4.

[56] Notice the symbol "$o$", probably used so as to allude to the fact that the quantity so denoted is vanishing (unfortunately, in [8], p. 48, this symbol is replaced with "0").

[57] A proof of that, if it were necessary, is that Archimedes' quadrature of the parabola is just the crucial result on which Barrow's proof of his first theorem is founded.

*Mutatis mutandis*, this is the same philosophical point that Barrow's first example seems to be concerned with: where potential infinity by division is legitimate, actual infinity by addition is not, or better, the latter becomes legitimate only if a model for it is provided, with respect to which it appears to be merely an expression of a potential infinity by division. This last point is explicitly made by Barrow himself, with reference to arithmetical series, in his ninth lecture[58]:

> Nay it is plainly taught and demonstrated by Arithmeticians, that an infinite Series of Fractions decreasing in a certain Proportion, is equal to a certain Number, or to Unite [*sic*], or to a part of an Unity; *ex. gr.* that such a Series of Fractions decreasing in a *subsesquialter Proportion* is equal to *Two*, in *subduble Proportion* to *Unity*, in a *subtriple* to *one half*; from whence it is not inconsistent for something finite to contain in it an Infinity of Parts; especially since nothing agrees with Number, which does not with more right with Magnitude, which Number represents and denominates.

As Barrow's reference to arithmeticians reveals, the opposition between this potential conception of infinity and an actual one has nothing to do, as such, with the subsumption of the domain of arithmetic under that of geometry.

### 3.4 Two last proofs

My argument could proceed further, by mentioning several other non explicitly geometric proofs of Barrow's theorem that, though possibly not known to him in 1664, became so later, or would have been easily understood by him. I shall limit myself to two of them.

The first one—of which Barrow certainly became aware in the second half of the sixties—depends on the so-called "method of division of Mercator". This name depends on the fact that the first printed source where it is exposed is Mercator's *Logarithmotechnia*, published in 1668[59]. Still, Newton used it, in 1664 and 1665, in the context of his own reformulation of Wallis' method of quadrature[60].

The proposition 2 included in a note that Newton wrote in Cambridge, in the winter 1664-1665, after his second reading of Wallis's *Arithmetica infinitorm*[61], consists of the following equality:

$$\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{bb} + \frac{aaxx}{b^3} - \frac{aax^3}{b^4} + \frac{aax^4}{b^5} - \frac{aax^5}{b^6} + \frac{aax^6}{b^7} + \&c. \tag{25}$$

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[58] Cf. [7], p. 157.

[59] Cf. [25], par. XV, pp. 29-30.

[60] Cf. [28], ch. 4.

[61] Cf. [27], vol. 1, **1**, 3, § 4, [2], pp. 126-134, specially p. 127.

This is the first statement of a particular case of what would become, quite soon, Newton's binomial theorem. The proof of this equality offered by Newton on such an occasion cannot be reconstructed here. I shall limit myself to remark that it is far from being plain, simple and compact, and is based on a quite conjectural extension of Pascal's triangle or Wallis's table of figurate numbers. Still, Newton comes back to this matter a few months later, presumably at the end of summer 1665, in a sketch of a treatise on quadratures and series expansions[62], whose propositions 7 and 8 consist of the following equalities:

$$\begin{aligned} \overline{a+b}\big\}^{\frac{m}{n}} \quad &= \quad a^{\frac{m}{n}} + \frac{m}{n}\times\frac{b}{a}\times a^{\frac{m}{n}} + \frac{m}{n}\times\frac{m-n}{2n}\times\frac{bb}{aa}\times a^{\frac{m}{n}} + \\ &\qquad + \frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{b^3}{a^3}\times a^{\frac{m}{n}}.\ \&c.; \\ \frac{1}{\overline{a+b}\big\}^{\frac{m}{n}}} \quad &= \quad \frac{1}{a^{\frac{m}{n}}} - \frac{m}{n}\times\frac{b}{a}\times\frac{1}{a^{\frac{m}{n}}} - \frac{m}{n}\times\frac{-m-n}{2n}\times\frac{bb}{aa}\times\frac{1}{a^{\frac{m}{n}}} + \\ &\qquad - \frac{m}{n}\times\frac{-m-n}{2n}\times\frac{-m-2n}{3n}\times\frac{b^3}{a^3}\times\frac{1}{a^{\frac{m}{n}}}.\ \&c. \end{aligned}$$

These equalities represent the first general statement of the binomial theorem[63]. Among other arguments that Newton offers for justifying them—including a quite complex one based, once again, on Pascal's triangle or Wallis's table of figurate numbers—we find the following[64]:

> The truth of these two prop: is also thus demonstrated. If $\overline{a+b}\big\}^{\frac{-1}{1}} = \frac{1}{a+b}$ I divide 1 by $a+b$ as in decimall fractions & find the quote $\frac{1}{a} - \frac{b}{aa} + \frac{bb}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5}$&*c*. as appeareth also by multiplying both parts by $a+b$. So I extract the rote of $a^2+b$ as if they were decimall numbers & find $\sqrt{a^2+b} = a + \frac{b}{2a} - \frac{bb}{8a^3} + \frac{b^3}{16a^5}$&*c*, as also may appear by squareing both parts.

We are, of course, interested in the first of these arguments, specially concerned with the first of the previous equalities. It just relies on Mercator's method. Its idea is simple: just apply to the binomial $a+b$ the usual method to divide 1 by any other number expressed as a sum of two numbers; first, divide 1 by $a$, multiply the result for $a+b$ and subtract the product from 1, so to get the first remainder, then operate on this remainder as on 1, and continue in this way as far as you like. If one note by "$A_i$" and "$R_i$" $(i = 1, 2, \ldots)$ the successive terms of the development of $\frac{1}{a+b}$ and the respective remainders, and take $R_1$ to be 1, one easily gets:

$$A_i = \frac{R_i}{a} \quad \text{and} \quad R_{i+1} = R_i - A_i\,(a+b)\,,$$

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[62] Cf. [27], vol. 1, 2, 5, § 1, [6], pp. 318-321, specially p. 321.

[63] Cf. [27], vol. 1, p. 321, note (111).

[64] Cf. [27], vol. 1, **2**, 5, § 1, [6], p. 321.

that is

$$\begin{array}{rcl} A_1 = \dfrac{R_1}{a} = \dfrac{1}{a} & ; & R_2 = R_1 - A_1\,(a+b) = -\dfrac{b}{a} \\ A_2 = \dfrac{R_2}{a} = -\dfrac{b}{a^2} & ; & R_3 = R_2 - A_2\,(a+b) = \dfrac{b^2}{a^2} \\ A_3 = \dfrac{R_3}{a} = \dfrac{b^2}{a^3} & ; & R_4 = R_3 - A_3\,(a+b) = -\dfrac{b^3}{a^3} \\ \ldots & & \ldots \\ A_n = \dfrac{R_n}{a} = -(-)^n\dfrac{b^{n-1}}{a^n} & ; & R_{n+1} = (-)^n\dfrac{b^n}{a^n} \\ \ldots & & \ldots, \end{array}$$

or

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \ldots - (-)^i\frac{b^{i-1}}{a^i} + \ldots \qquad (26)$$

To get the equality (25) it is then enough to operate appropriate substitutions and multiplications. In the same way, by replacing, in the equality (26), $a$ with $\alpha$ and $b$ with $-\beta$, then by multiplying for $\alpha^2$, one gets the equality (8), that is thus proved through an argument whose arithmetical origins are evident.

If Barrow did not know this argument in 1664, he certainly became aware of it later on, since Newton exposed it, in a quite general form, in the *De analysi*[65], a text that Barrow himself sent to Collins on July 31st, 1669[66].

Another possible, non explicitly geometric, proof of Barrow's theorem could be done by relying on the so-called method of indeterminate coefficients. This method was exposed, in its finitary form, in Descartes' *Geometry*[67], where it was used to solve the problem of tangents. Later on, and especially after its massive use by Newton, the infinitary version of this method became a crucial tool of the analysis. Such a method can be quite easily applied to the problem of looking for the sum of $\frac{\alpha^2}{\alpha-\beta}$. Suppose that:

$$\frac{\alpha^2}{\alpha-\beta} = A + B\beta + C\beta^2 + D\beta^3 + \ldots,$$

where $A$, $B$, $C$, $D$, ... are indeterminate coefficients to be determined. Then

$$\begin{array}{rcl} \alpha^2 & = & \big(A + B\beta + C\beta^2 + D\beta^3 + \ldots\big)\,(\alpha-\beta) \\ & = & A\alpha + B\alpha\beta + C\alpha\beta^2 + D\alpha\beta^3 + \ldots - A\beta - B\beta^2 - C\beta^3 - D\beta^4 - \ldots, \end{array}$$

and thus, by separately equating the coefficients of the successive powers of $\beta$ to zero:

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[65] Cf. [27], vol. II, **2**, 3, pp. 212-214.

[66] Cf. [26], vol. I, p. 14 and [27], vol. II, **2**, *Introduction*, note (11), p. 166.

[67] Cf. [10], p. 347.

$$
\begin{array}{rcl}
\alpha^2 - A\alpha = 0 & \rightarrow & A = \alpha, \\
A - B\alpha = 0 & \rightarrow & B = \frac{A}{\alpha} = 1, \\
B - C\,\alpha = 0 & \rightarrow & C = \frac{B}{\alpha} = \frac{1}{\alpha}, \\
C - D\alpha = 0 & \rightarrow & D = \frac{C}{\alpha} = \frac{1}{\alpha^2}, \\
\dots & & \dots,
\end{array}
$$

according to the equality (15).

Both these proofs have a double disadvantage compared with on Viète's and Barrow's: they require a previous conjecture about the sum of the geometric series, that is, they have no heuristic power; and they do not provide any argument for warranting the infinite reiteration of the procedure on which they depend. The second one supposes, moreover, that the ratio $\frac{\alpha^2}{\alpha-\beta}$ has a unique expansion in a power series and that this power series can be manipulated term by term. If Barrow had known them when he gave his *Mathematical Lectures*, he could have advanced these reasons to prefer his own proof to them. But, if so, he would have implicitly opposed an ideal of perspicuity of mathematical arguments to an ideal of extension of mathematical techniques, rather than showing any reason for the domain of arithmetic to be subsumed under that of geometry.

## 4 Conclusions

The consideration of Barrow's examples should show that, when compared with the actual state of mathematical research in the middle of the 17th century, his opposition of geometry and arithmetic is out of place. These examples could have only been used in order to support the thesis according to which only a geometrical interpretation of infitary arguments would be able to license these same arguments, since, as Aristotle had already argued, the only legitimate sort of infinity would be the infinity of the division of an already given magnitude.

Still, the second example, namely the proof that Barrow offered for the theorem with which this example is concerned, possibly had an unexpected influence on the evolution of geometry. In a note he wrote at the time of his priority quarrel with Leibniz, probably in 1714, Newton claimed that "Dr Barrows Lectures might put [...][him] upon considering the generation of figures by motion[68]". Though he immediately added that he was not able to remember it exactly, this statement should be enough to attest that Newton followed at least some of Barrow's *Mathematical Lectures*, namely the parts of them dealing with a kinematic approach to geometry, and thus, quite probably, the lecture where Barrow exposed his second example.

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[68] Cf. [27], vol. 1, **2**, 5, § 4, footnote (4), p. 344. Cf. also: *ibid.*, vol. 1, **2**, *Introduction*, App. 1, p. 150.

It is thus possible to conjecture that this example had some influence on the development of Newton's ideas about geometry. This is in any case what Hofmann has suggested[69].

In my view, a crucial idea that Newton could have drawn from Barrow's proof is that of associating to a pair of segments, supposed to be generated in equal time by a moving point, the ratio of the speeds of the generating motions. This is more than merely admitting that a segment is generated through motion, which was quite a widespread conception in the middle of the 17th century. It rather entails that the speed of this motion is an essential feature to be considered in geometry. This is no more a philosophical option. It is rather a view on the intrinsic mathematical nature of geometric objects and their relations that constituted one of the crucial elements on which Newton's theory of fluxions was founded.

## 5 Annex 1: An arithmetical proof of Barrow's first theorem through a result of Ramanujan

Ramanujan's result is the following:

$$\sum_{i=1}^{n} \sqrt{i} = C + \left[\frac{2}{3}n + \frac{1}{2}\right]\sqrt{n} + \frac{1}{6}\sum_{j=0}^{\infty}\left[\sqrt{n+j} + \sqrt{n+j+1}\right]^{-3}, \tag{27}$$

where $C$ is an appropriate constant.

To prove it, Ramanujan relies on the definition of the following function $\phi : \mathbb{N}^{+} \rightarrow \mathbb{R}$:

$$\phi(n) = \sum_{i=1}^{n} \sqrt{i} - \left(C + \frac{2}{3}n\sqrt{n} + \frac{1}{2}\sqrt{n}\right) - \frac{1}{6}\sum_{j=0}^{\infty}\left[\sqrt{n+j} + \sqrt{n+j+1}\right]^{-3}, \tag{28}$$

where $C$ is such that

$$\phi(1) = 1 - C - \frac{2}{3} - \frac{1}{2} - \frac{1}{6}\sum_{j=0}^{\infty}\left[\sqrt{1+j} + \sqrt{2+j}\right]^{-3} = 0. \tag{29}$$

This is a finite constant, since

$$\sum_{j=0}^{\infty}\left[\sqrt{1+j} + \sqrt{2+j}\right]^{-3}$$

converges.

---

[69] Cf. [16], p. 115. Notice however that Whiteside has remarked that Newton could have also drawn his ideas about the generation of geometric objects by motion from some other sources [cf. [27], vol. 1, **2**, 5, § 4, footnote (4), p. 344]. Possibly, Newton's sources were many, but Barrow's example was probably one of them.

Once the function (28) is defined, the proof reduces to a quite easy calculation that Barrow could have performed himself. Thus, its difficulty consists in imagining of the appropriate form of the function $\phi(n)$: only the extraordinary numerical sense of someone like Ramanujan could have had this insight.

The first step in this calculation is the establishment of the following equality:

$$\phi(n)-\phi(n+1)=-\sqrt{n+1}+\frac{2}{3}\left[(n+1)\sqrt{n+1}-n\sqrt{n}\right]+\frac{1}{2}\left[\sqrt{n+1}-\sqrt{n}\right]+\frac{1}{6}\left[\sqrt{n}-\sqrt{n+1}\right]^3. \tag{30}$$

This is an immediate consequence of the other equality

$$\frac{1}{(a+b)^\nu}=\frac{(a-b)^\nu}{(a^2-b^2)^\nu},$$

that holds for any $\nu$, $a$ and $b$. It is enough, indeed, to put $\nu=3$, $a=\sqrt{n}$ and $b=\sqrt{n+1}$, for getting:

$$\frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)^3}=-\left(\sqrt{n}-\sqrt{n+1}\right)^3. \tag{31}$$

From (30), it is then easy to draw, by a simple calculation, that

$$\phi(n)-\phi(n+1)=0.$$

And, as $\phi(1)=0$, this entails that $\phi(n)=0$, for any (positive natural) $n$, which is equivalent to the equality (1)[70].

Once this last equality is admitted, to prove Barrow's first theorem it is enough to remark that from it and the other equality (31), it follows that

$$\frac{\sum_{i=1}^{n}\sqrt{i}}{(n+1)\sqrt{n}}=\frac{C}{(n+1)\sqrt{n}}+\frac{4n+3}{6(n+1)}-\frac{1}{6}\sum_{j=0}^{\infty}\frac{\left[\sqrt{n+j}-\sqrt{n+j+1}\right]^3}{(n+1)\sqrt{n}}$$

---

[70] Let us note that from (29) and (31), it follows that:

$$\begin{aligned}
C &= -\frac{1}{6}\left[1-\sum_{j=0}^{\infty}\left[\sqrt{1+j}-\sqrt{2+j}\right]^3\right]\\
&= -\frac{1}{6}\left[1-\left[\sqrt{1}-\sqrt{2}\right]^3-\left[\sqrt{2}-\sqrt{3}\right]^3-\left[\sqrt{3}-\sqrt{4}\right]^3-\ldots\right]\\
&= -\frac{1}{6}\left[1-\left[7-5\sqrt{2}\right]-\left[11\sqrt{2}-9\sqrt{3}\right]-\left[15\sqrt{3}-13\sqrt{4}\right]-\ldots\right]\\
&= 1+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}-\frac{1+4n}{6}\sqrt{n+1}+\ldots
\end{aligned}$$

and thus[71]:

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sum\limits_{i=0}^{n}\sqrt{i}}{(n+1)\sqrt{n}} &= \lim_{n\to\infty}\left[\frac{C}{(n+1)\sqrt{n}}+\frac{4n+3}{6(n+1)}-\frac{1}{6}\sum_{j=0}^{\infty}\frac{\left[\sqrt{n+j}-\sqrt{n+j+1}\right]^{3}}{(n+1)\sqrt{n}}\right] \\
&= \lim_{n\to\infty}\frac{C}{(n+1)\sqrt{n}}+\lim_{n\to\infty}\frac{4n+3}{6(n+1)}-\sum_{j=0}^{\infty}\lim_{n\to\infty}\frac{\left[\sqrt{n+j}-\sqrt{n+j+1}\right]^{3}}{6(n+1)\sqrt{n}} \\
&= 0+\frac{2}{3}-0-0-0-\ldots=\frac{2}{3}.
\end{aligned}$$

# 6 Annex 2: A proof of Barrow's first theorem through integration

According to the Cauchy-Riemann definition of a definite integral, one gets:

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\varepsilon\sqrt{i\varepsilon}=\int_{0}^{1}\sqrt{x}dx.$$

If one supposes that $\varepsilon=\frac{1}{n}$, it follows that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{i}}{n\sqrt{n}}=\int_{0}^{1}\sqrt{x}dx.$$

As

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{i}}{n\sqrt{n}}=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{\sqrt{i}}{n\sqrt{n}}=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{\sqrt{i}}{(n+1)\sqrt{n}},$$

this is the same as:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{\sqrt{i}}{(n+1)\sqrt{n}}=\int_{0}^{1}\sqrt{x}dx.$$

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[71] From the binomial theorem it follows, for any natural $j$, that

$$\begin{aligned}
\frac{\left[\sqrt{n+j}-\sqrt{n+j+1}\right]^{3}}{(n+1)\sqrt{n}} &= \frac{[4n+4j+3]\sqrt{n+j}-[4n+4j+1]\sqrt{n+j+1}}{(n+1)\sqrt{n}} \\
&= \frac{[4n+4j+3]\sum\limits_{i=0}^{\infty}\begin{pmatrix}1/2\\ i\end{pmatrix}\left(\sqrt{n}\right)^{1-2i}j^{i}-[4n+4j+1]\sum\limits_{i=0}^{\infty}\begin{pmatrix}1/2\\ i\end{pmatrix}\left(\sqrt{n}\right)^{1-2i}(j+1)^{i}}{(n+1)\sqrt{n}}
\end{aligned}$$

and so:

$$\lim_{n\to\infty}\frac{\left[\sqrt{n+j}-\sqrt{n+j+1}\right]^{3}}{(n+1)\sqrt{n}}=\lim_{n\to\infty}\frac{2\sqrt{n}}{(n+1)\sqrt{n}}=\lim_{n\to\infty}\frac{2}{n+1}=0.$$

To prove the equality (1), it is thus enough to remark that

$$\int_0^1 \sqrt{x}dx = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}.$$

Notice that the limits $\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\varepsilon\sqrt{i\varepsilon}$, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}}$, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{i}}{n\sqrt{n}}$, and $\lim_{n\to\infty}\sum_{i=0}^{n}\frac{\sqrt{i}}{(n+1)\sqrt{n}}$ that enter this argument are essentially different from the limit $\lim_{n\to\infty}\frac{C}{(n+1)\sqrt{n}}$, $\lim_{n\to\infty}\frac{4n+3}{6(n+1)}$, and $\lim_{n\to\infty}\frac{1}{6}\sum_{j=0}^{\infty}\frac{\left[\sqrt{n+j}-\sqrt{n+j+1}\right]^3}{(n+1)\sqrt{n}}$ that enter Ramanujan's argument, since the former cannot commute with $\sum$. In the former, this last operator refers, indeed, to the same variable as the limit operator or to a function of this variable. This is a typical feature of the Cauchy-Riemann integral, and it is because of it that the argument based on such an integral is not purely arithmetic, while Ramanujan's one is so.

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# *EUCLIDE*'s ELEMENTS;

The whole

# FIFTEEN BOOKS,

compendiouſly Demonſtrated:

WITH

## *ARCHIMEDES*'s Theorems of the Sphere and Cylinder, Inveſtigated by the *Method of Indiviſibles.*

ALSO,

# *EUCLIDE*'s DATA,

and a brief

## Treatiſe of REGULAR SOLIDS.

---

By ISAAC BARROW, D.D. *late Maſter of* Trinity *College in* Cambridge.

---

The whole carefully Corrected, and Illuſtrated
with Copper Plates.

To which is now added an

# APPENDIX,

Containing,

## The Nature, Conſtruction, and Application of Logarithms.

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By J. BARROW, Author of *Navigatio Britannica*, &c.

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*LONDON*: Printed for W. and J. Mount, and T. Page
on *Tower-Hill*; and C. Hitch and L. Hawes in *Pater-noſter-Row*; R. Manby and S. Cox on *Ludgate-Hill*; E. Comyns
under the *Royal-Exchange*; J. and J. Rivington in St. *Paul's*
*Church-Yard*; and J. Ward in *Cornhill*, oppoſite the *Royal-Exchange*, 1751.

# To the READER.

*IF you are deſirous, Courteous Reader, to know what I have performed in this Edition of the* ELEMENTS of EUCLIDE, *I ſhall here explain it to you in ſhort, according to the Nature of the Work. I have endeavoured to attain two Ends chiefly; the firſt, to be very perſpicuous, and at the ſame time ſo very brief, that the Book may not ſwell to ſuch a Bulk, as may be troubleſome to carry about one, in both which I think I have ſucceeded. Some of a brighter Genius, and endued with greater Skill, might have demonſtrated moſt of theſe Propoſitions with more nicety, but perhaps none with more ſuccinctneſs than I have; eſpecially ſince I alter'd nothing in the Number and Order of the Author's Propoſitions; nor preſum'd either to take the Liberty of rejecting, as leſs neceſſary, any of them, or of reducing ſome of the eaſier ſort into the Rank of Axioms, as ſeveral have done; and among others, that moſt expert Geometrician* A. Tacquetus C. *(whom I the more willingly name, becauſe I think it is but civil to acknowledge that I have imitated him in ſome Points) after whoſe moſt accurate Edition I had no Thoughts of attempting any thing of this Nature, 'till I conſidered that this moſt learned Man thought fit to publiſh only Eight of* EUCLIDE'*s Books, which he took the pains to explain and embelliſh, having in a manner rejected and undervalued the other Seven, as leſs appertaining to the Elements of Geometry. But my Province was originally quite different, not that of writing the Elements of Geometry after what method ſoever I pleas'd, but of demonſtrating, in as few Words as poſſible I could, the whole Works of* EUCLIDE. *As* *to Four of the Books,* viz. *the Seventh, Eighth, Ninth, and Tenth, although they do not ſo nearly appertain to the Elements of plain and ſolid Geometry, as the ſix precedent and the two ſubſequent, yet none of the more ſkilful Geometricians can be ſo ignorant as not to know that they are very uſeful for Geometrical Matters, not only by reaſon of the very near affinity that is between Arithmetick and Geometry, but alſo for the Knowledge of both commenſurable and incommenſurable Magnitudes, ſo exceeding neceſſary for the Doctrine of both plain and ſolid Figures. Now the noble Contemplation of the five regular Bodies that is contained in the three laſt Books, cannot, without great Injuſtice, be pretermitted, ſince that for the ſake thereof our* ϛοιχειώτής, *being a Philoſopher of the* Platonic Sect, *is ſaid to have compos'd this univerſal Syſtem of Elements; as* Proclus *lib.* 2. *witneſſeth in theſe Words,* Ὅθεν δὴ ϗ τῆς συμπάσης ϛοιχειώσεως τέλος προεϛήσατο τὴν τῶν καλȣμένων πλατωνικῶν χημάτων σύϛασιν. *Beſides, I eaſily perſuaded my ſelf to think, that it would not be unacceptable to any Lover of theſe Sciences to have in his Poſſeſſion the whole* Euclidean *Work, as it is commonly cited and celebrated by all Men: Wherefore I reſolved to omit no Book or Propoſition of thoſe that are found in* P. Herigonius's *Edition, whoſe Steps I was obliged cloſely to follow, by reaſon I took a Reſolution to make uſe of moſt of the Schemes of the ſaid Book, very well foreſeeing that Time would not allow me to form new ones, though ſometimes I choſe rather to do it. For the ſame Reaſon I was willing to uſe for the moſt part* EUCLIDE's *own Demonſtrations, having only expreſs'd them in a more ſuccinct Form, unleſs perhaps in the Second, Thirteenth, and very few in the Seventh, Eighth, and Ninth Books, in which it ſeem'd not worth my while to deviate in any Particular from him: Therefore I am not* *without great hopes, that as to this Part I have in ſome meaſure ſatisfied both my own Intentions, and the Deſire of the Studious. As for ſome certain Problems and Theorems that are added in the Scholions (or ſhort Expoſitions) either appertaining (by reaſon of their frequent Uſe) to the Nature of theſe Elements, or conducing to the ready Demonſtration of thoſe Things that follow, or which imitate the Reaſons of ſome principal Rules of Practical Geometry, reducing them to their original Fountains, theſe I ſay, will not, I hope, make the Book ſwell to a Size beyond the deſign'd Proportion.*

*The other Butt which I levell'd at, is to content the Deſires of thoſe who are delighted more with ſymbolical than verbal Demonſtrations. In which Kind, whereas moſt among us are accuſtom'd to the Symbols of* Gulielmus Oughtredus, *I therefore thought beſt to make uſe, for the moſt part, of his. None hitherto (as I know of) has attempted to interpret and publiſh* EUCLIDE *after this manner, except* P. Herigonius; *whoſe Method (tho' indeed moſt excellent in many things, and very well accommodated for the particular purpoſe of that moſt ingenious Man) yet it ſeems in my Opinion to labour under a double Defect. Firſt, in regard, that, altho' of two or more Propoſitions produced for the Proof of any one Problem or Theorem, the former do not always depend on the latter, yet it do not readily enough appear, either from the order of each, or by any other manner, when they agree together, and when not; wherefore, for want of the Conjunctions and Adjectives,* ergo, rurſus, *&c. many difficulties and occaſions of doubt often ariſe in reading, eſpecially to thoſe that are Novices. Beſides it frequently happens, that the ſaid Method cannot avoid ſuperfluous Repititions, by which the Demonſtrations are oftentimes*

*times render'd tedious, and ſometimes alſo more intricate; which Faults my Method eaſily removes by the arbritary mixture of both Words and Signs: Therefore let what has been ſaid, touching the Intention and Method of this little Work, ſuffice. As to the reſt, whoever covets to pleaſe himſelf with what may be ſaid, either in Praiſe of the Mathematicks in general, or of Geometry in particular, or touching the Hiſtory of theſe Sciences, and conſequently of* EUCLIDE *himſelf, (who digeſted thoſe Elements) and others* ἐξωτερικοὶ *of that kind, may conſult other Interpreters. Neither will I (as if I were afraid leſt theſe my Endeavours may fall ſhort of being ſatisfactory to all Perſons) alledge as an Excuſe (though I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occaſioned by other Affairs, nor yet the want of requiſite help for theſe Studies, nor ſeveral other things of the like Nature. But what I have here employ'd my Labour and Study in for the Uſe of the ingenious Reader, I wholly ſubmit to his Cenſure and Judgment, to approve if uſeful, or reject if otherwiſe.*

I. B.

# To the READER.

*Thus far the Learned Author.* —— *But as the work has been often printed ſince his death, and by that means ſeveral errors committed, I have, at the requeſt of the Bookſellers concerned in this treatiſe, and from a ſincere reſpect and veneration for the memory of the deceaſed Author, carefully reviſed the whole performance; and flatter myſelf, from the great pains and care I have taken, that very few errors will be found in this Edition.*

*And as the wooden cuts in the former editions, were, by often printing, almoſt obliterated, their place is now ſupplied by figures engraven on Copper Plates, and proper care taken to correct the inaccuracies and errors committed by cutting them on Wood; which has given this edition great advantages over the former, both with regard to beauty and correctneſs.*

*In the* Appendix *which I have added to this work, I have endeavoured to render the conſtruction and uſe of Logarithms as plain and eaſy as poſſible. And that nothing might be wanting for underſtanding the nature of thoſe tables uſed in trigonometrical calculations, I have added an inveſtigation of the ſeveral ſeries invented by the illuſtrious Sir* Iſaac Newton, *for finding the length of the circumference of the circle, in equal parts of the radius, alſo of the ſine, tangent and ſecant of any arch, in the ſame parts; with the application of theſe ſeries to the conſtructing the triangular canon, and the quadrature of the Circle. I have alſo ſhewn the manner of computing the artificial or logarithmic ſines, tangents, and ſecants, from the length of the arch of the circle firſt given in equal parts of the radius, independant of the tables of logarithms, ſines, tangents, and ſecants.*

J. Barrow.

# The Explication of the Signs or Characters.

| | | |
|---|---|---|
| $=$ | Signifies | Equal. |
| $\sqsubset$ | | Greater. |
| $\sqsupset$ | | Leſſer. |
| $+$ | | More, or to be added. |
| $-$ | | Leſs, or to be ſubtracted. |
| $-:$ | | The Differences, or Exceſs; Alſo, that all the Quantities which follow, are to be ſubtracted, the Signs not being changed. |
| $\times$ | | Multiplication, or the Drawing one ſide of a Rectangle into another.<br>The ſame is denoted by the Conjunction of Letters; as $AB = A \times B$. |
| ∺ | | Continued Proportion. |
| $\surd$ | | The Side or Root of a Square, or Cube, &c. |
| Q & q | | A Square. |
| C & c | | A Cube. |
| Q. Q. | | The Ratio of a ſquare Number to a ſquare Number. |

*Other Abbreviations of Words, where-ever they occur, the Reader will without trouble underſtand of himſelf; ſaving ſome few, which, being of leſs general uſe, we refer to be explained in their Places, moſt commonly at the beginning of each Book in which they are uſed.*

# The FIRST BOOK OF EUCLIDE's ELEMENTS.

## *Definitions.*

I. A Point is that which hath no parts.

II. A Line is a longitude without latitude.

III. The ends, or limits of a line are points.

IV. A Right-line is that which lies equally betwixt its points.

V. A Superficies is that which hath only longitude and latitude.

VI. The extremes, or limits of a ſuperficies are lines.

VII. A Plain-ſuperficies is that which lies equally betwixt its lines.

VIII. A Plain-angle is the inclination of two lines the one to the other, the one touching the other in the ſame plain, yet not lying in the ſame ſtrait-line.

IX. And if the lines which contain the angle, be right-lines, it is called a right-lined angle.

Plate I. *Fig.* 1. X. When a right-line CG, ſtanding upon a right-line AB, makes the angles on either ſide thereof, CGA, CGB, equal one to the other, then both thoſe equal angles are right-angles; and the right-line CG, which ſtandeth on the other, is termed a Perpendicular to that (AB) whereon it ſtandeth.

Note, *When ſeveral angles meet at the ſame point* (*as at* G) *each particular angle is deſcribed by three letters; whereof the middle letter ſheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right-lines* CG, AG *make at* G, *is called* CGA, *or* AGC.

XI. An Obtuſe-angle is that which is greater than a right-angle; as DGB.

XII. An Acute-angle is that which is leſs than a right-angle; as DGA.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

*Fig.* 2. XV. A Circle is a plain figure contained under one line, which is called a circumference; unto which all lines, drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.

XVI. And that point is called the center of the circle.

XVII. A Diameter of a circle is a right-line drawn thro' the center thereof, and ending at the circumference on either ſide, dividing the circle into two equal parts.

XVIII. A Semicircle is a figure which is contained under the diameter and that part of the circumference which is cut off by the diameter.

*In the circle* EABCD, E *is the center,* AC *the diameter,* ABC *the ſemicircle.*

XIX. Right-lined figures are ſuch as are contained under right-lines.

XX. Three-ſided or trilateral figures are ſuch as are contained under three right-lines.

XXI. Four-ſided or quadrilateral figures are ſuch as are contained under four right-lines.

XXII. Many-ſided figures are ſuch as are contained under more right-lines than four.

XXIII. Of trilateral figures, that is, an equilateral triangle, which hath three equal ſides; as the triangle ABC. *Fig.* 3.

XXIV. An Iſoſceles, is a triangle which hath only two ſides equal; as the triangle ABC. Plate I. *Fig.* 4.

XXV. A Scalenum, is a triangle whoſe three ſides are all unequal; as ABC. *Fig.* 5.

XXVI. Of theſe trilateral figures, a right-angled triangle is that which hath one right-angle; as the triangle ABD. *Fig.* 5.

XXVII. An Amblygonium, or obtuſe-angle triangle, that is which hath one angle obtuſe; as ABC. *Fig.* 5.

XXVIII An Oxgonium, or acute-angled triangle, is that which hath three acute angles; as ABC. *Fig.* 4.

An equiangular, or equal-angled figure is that whereof all the angles are equal. Two figures are equiangular, if the ſeveral angles of the one figure be equal to the ſeveral angles of the other. The ſame is to be underſtood of equilateral figures.

XXIX. Of Quadrilateral, or four-ſided figures, a ſquare is that whoſe ſides are equal, and angles right; as ABCD. *Fig.* 6.

XXX. A Figure on the one part longer, or a long ſquare, is that which hath right-angles, but not equal ſides; as ABCD. *Fig.* 7.

XXXI. A Rhombus, or diamond-figure, is that which has four equal ſides, but is not right-angled; as ABCD. *Fig.* 8.

XXXII. A Rhomboides, is that whoſe oppoſite ſides, and oppoſite angles are equal; but has neither equal nor right angles; as ABCD. *Fig.* 9.

XXXIII. All other quadrilateral figures beſides theſe are called *trapezia*, or tables; as GNDH. *Fig.* 10.

XXXIV. Parallel, or equi-diſtant right-lines are ſuch, which being in the ſame ſuperficies, if infinitely produced, would never meet as AD and BC. *Fig.* 11.

XXXV. A Parallelogram is a quadrilateral figure, whoſe oppoſite ſides are parallel, or equi-diſtant; as ABCD. Plate I. *Fig.* 7.

XXXVI. In a Parallelogram AGEL, *Fig.* 9. when a diameter AE, and two lines BK, CF, parallel to the ſides, cutting the diameter in one and the ſame point D, are drawn, ſo that the Parallelogram be divided by them into four Parallelograms; thoſe two LD, DG, through which the diameter does not paſs, are called complements; and the other two CB, KF, through which the diameter paſſeth, the Parallelograms ſtanding about the diameter.

A Problem *is, when ſomething is propoſed to be done or effected.*

A Theorem *is, when ſomething is propoſed to be demonſtrated.*

A Corollary *is a Conſectary, or ſome conſequent truth gained from a preceding demonſtration.*

A Lemma *is the demonſtration of ſome premiſe, whereby the proof of the thing in hand becomes the ſhorter.*

## *Poſtulates or Petitions.*

1. FRom any given point to any other given point, to draw a right-line.

2. To produce a finite right-line, ſtrait forward continually.

3. Upon any center, and at any diſtance, to deſcribe a circle.

## *Axioms.*

1. THings equal to the ſame thing, are alſo equal one to the other.

As $A=B=C$. Therefore $A=C$; or therefore all A,B,C, are equal the one to the other.

Note, *When ſeveral quantities are joyned the one to the other continually with this mark* $=$, *the firſt quantity is by virtue of this axiom equal to the laſt, and every one to every one: In which caſe we often abſtain from citing the axiom, for brevity's ſake; altho' the force of the conſequence depends thereon.*

2. If to equal things you add equal things, the wholes will be equal.

3. If from equal things you take away equal things, the things remaining will be equal.

4. If to unequal things you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the ſame third, or to equal things, are equal one to the other. Underſtand the ſame of triple, quadruple, *&c.*

7. Things which are half of one and the ſame thing, or of things equal, are equal the one to the other. Conceive the ſame of ſubtriple, ſubquadruple, *&c.*

8. Things which agree together, are equal one to the other.

*The converſe of this axiom is true in right-lines and angles, but not in figures, unleſs they be like.*

*Moreover, magnitudes are ſaid to agree, when the parts of the one being apply'd to the parts of the other, they fill up an equal or the ſame place.*

9. Every whole is greater than its part.

10. Two right-lines cannot have one and the ſame ſegment (or part) common to them both.

11. Two right-lines meeting in the ſame point, if they be both produced, they ſhall neceſſarily cut one another in that point.

12. All right-angles are equal the one to the other.

13. If a right-line EF (*Plate* 1. *Fig.* 11.) falling on two right-lines, AD, BC, make the internal angles on the ſame ſide, GFE, FEG, leſs than two right-angles, thoſe two right-lines produced ſhall meet on that ſide where the angles are leſs than two right-angles.

14. Two right-lines do not contain a ſpace.

15. If to equal things you add things unequal, the exceſs of the wholes ſhall be equal to the exceſs of the additions.

16. If to unequal things equal be added, the exceſs of the wholes ſhall be equal to the exceſs of thoſe which were at firſt.

17. If from equal things unequal things be taken away, the exceſs of the remainders ſhall be equal to the exceſs of what was taken away.

18. If from things unequal things equal be taken away, the exceſs of the remainders ſhall be equal to the exceſs of the wholes.

19. Every whole is equal to all its parts taken together.

20. If one whole be double to another, and that which is taken away from the firſt be double to that which is taken away from the ſecond, the remainder of the firſt ſhall be double to the remainder of the ſecond.

*The Citations are to be underſtood in this manner; When you meet with two numbers, the firſt ſhews the Propoſition, the ſecond the Book.*; as by 4. 1. you are to underſtand the fourth Propoſition of the firſt Book; and ſo of the reſt.

*Moreover, ax. denotes Axiom, poſt. Poſtulate, def. Definition, ſch. Scholium, cor. Corollary.*

## PROPOSITION I. Plate I. *Fig.* 12.

U*Pon a finite right-line given as* AB, *to deſcribe an equilateral triangle* ACB.

a 3. *poſt.* From the centers A and B, at the diſtance of AB,
b 1. *poſt.* or BA, *(a)* deſcribe two circles cutting each other in
c 15. *def.* the point C; from whence *(b)* draw two right-lines
d 1. *ax.* CA, CB. Then is AC *(c)*=AB *(c)* = BC *(d)*=AC.
e 23. *def.* *(e)* Wherefore the triangle ACB is equilateral. *Which was to be done.*

### *Scholium.*

After the ſame manner upon the line AB may be deſcribed an Iſoſceles triangle, if the diſtances of the equal circles be taken greater or leſs than the line AB.

## PROP. II. *Fig.* 13.

*From a point given* A, *to draw a right-line* AG *equal to a right-line given* BC.

a 3. *poſt.* From the center C, at the diſtance of CB, *(a)* deſcribe
b 1. *poſt.* the circle CBE. *(b)* Join AC; upon which *(c)* raiſe the
c 1. 1. equilateral triangle ADC. *(d)* Produce DC to E. From
d 2. *poſt.* the center D, at the diſtance of DE, deſcribe the circle
e 2. *poſt.* DEG; and let DA *(e)* be produced to the point G in the
circumference thereof. Then AG=CB.

f 15. *def.* For DG *(f)*=DE, and DA *(g)* = DC. Wherefore
g *conſtr.* AG *(h)* = CE *(k)* = BC *(l)*. *Which was to be done.*
h 3 *ax.* The putting of the point A within or without the line
k 15. *def.* BC varies the caſes; but the conſtruction, and the de-
l. 1. *ax.* monſtration, are every where alike.

*Schol.*

The line AG might be taken with a pair of compaſſes; but the ſo doing anſwers to no poſtulate, as *Proclus* well obſerves.

### PROP. III. Plate I. *Fig.* 14.

*Two right-lines,* A *and* DC, *being given, from the greater* DC, *to take away the right-line* DE, *equal to the leſſer* A.

From the point D (*a*) draw the right-line BD = A. a 2. 1.
The circle deſcribed from the center D at the diſtance b 15. *df.*
of BD ſhall cut off DE (*b*) = BD (*c*) = A (*d*) = c *conſtr.*
DE. *Which was to be done.* d 1. *ax.*

### PROP. IV. *Fig.* 15, *and* 16.

*If two triangels* BAC, EDF, *have two ſides of the one* BA, AC, *equal to two ſides of the other* ED, DF, *each to its correſpondent ſide (that is* BA=ED, *and* AC=DF*) and have the angel* A *equal to the angle* D *contained under the equal right-lines; they ſhall have the baſe* BC *equal to the baſe* EF; *and the triangle* BAC *ſhall be equal to the triangle* EDF; *and the remaining angels* B, C, *ſhall be equal to the remaining angles,* E, F, *each to each, under which the equal ſides are ſubtended.*

If the point D be apply'd to the point A, and the right-line DE plac'd upon the right-line AB, the point E ſhall fall upon B, becauſe DE (*a*)=AB, alſo the right- a *hyp*
line DF ſhall fall upon AC, becauſe the angle A (*a*)=D. Moreover the point F ſhall fall upon the point C, becauſe AC (*a*)=DF. Therefore the right-lines EF, BC, ſhall agree, becauſe they have the ſame terms, and conſequently are equal. Wherefore the triangles, BAC, DEF, and the angles B, E, as alſo the angles C, F, do agree, and are equal. *Which was to be demonſtrated.*

### PROP. V. *Fig.* 4.

*The angles* BAC, BCA, *at the baſe of an Iſoſceles triangle* ABC, *are equal one to the other, And if the equal ſides* AB, BC, *are produced, the angles* CAD, ACE, *under the baſe, ſhall be equal one to the other.*

a 3. 1. (a) Take BE = BD; and (b) join CD, and AE.
b 1. *poſt.* Becauſe, in the triangles BCD, BAE, are AB (c) =
c *hyp.* BC, and BE, (d) = BD, and the angle B common to
d *conſtr.* them both, (e) therefore is the angle BAE = BCD,
e 4. 1. and the angle AEB (e) = BDC, and the baſe AE (e) =
f 3. *ax* CD; alſo EC (f) = DA. Therefore in the triangles AEC,
g 4. 1. ADC (g) will be the angle ECA = DAC. *Which was to
be dem.* Alſo the angle EAC = DCA, but the angle
h *before.* BAE (h) = BCD; therefore the angle BAC (k) =
k 3. *ax.* BCA. *Which was to be demonſtrated.*

*Coroll.*

Hence, every equilateral triangle is alſo equiangular.

### PROP. VI. Plate I *Fig.* 4.

*If two angles* BCA, ACB *of a triangle* ABC, *be equal the one to the other, the ſides* BC, AB, *ſubtended under the equal angles, ſhall alſo be equal one to the other.*

If the ſides be not equal, let one be bigger than the
a 3. 1. other, ſuppoſe BA ⊏ BC. (a) Make AF = BC, and (b)
b 1. *poſt.* draw the line CF.
c *ſuppoſ.* In the triangles FAC, BAC, becauſe AF (c) = CB, and
d *hyp.* the ſide AC is common, and the angle BAC (d) = ACB,
e 4. 1. the triangles FAC, ACB (e) ſhall be equal the one to the
f 9. *ax.* other, a part to the whole, (f) *Which is impoſſible.*

*Coroll.*

Hence, every equiangular triangle is alſo equilateral.

### PROP. VII. *Fig.* 4.

*Upon the ſame right-line* AC *two right-lines being drawn* AB, BC, *two other right-lines equal to the former,* AF, CF, *each to each* (viz. AF = AB, *and* CF = BC) *cannot be drawn from the ſame points* A, C, *on the ſame ſide* B, *to ſeveral points, as* B *and* F, *but only to* B.

1. *Caſe.* If the point F be ſet in the line AB, it is
a 9 *ax.* plain that AF is (a) not equal to AB.
2. *Caſe.* If the point D be placed within the triangle
*Fig.* 17. ACB, then draw the line CD, and produce BDF, and BCF.
b 5. 1. Now you would have AD = AC, then the angle ADC (b)
c *ſuppoſ.* = ACD; as alſo, becauſe BD (c) = BC, the angle FDC =

(b)

(b) ECD, therefore is the angle FDC⊏ (d) ACD, that d 9 *ax.*
is, the angle FDC⊏ADC. (d) *Which is impoſſible.*

3. *Caſe.* If D (*Fig.* 18.) falls without the triangle ACB, let CD be joined.

Again, the angle ACD (e) =ADC, and the angle BCD e 5. 1.
(c) =BDC. (f) Therefore the angle ACD⊏BDC, *viz.* f 9. *ax.*
the angle ADC⊏BDC. *Which is impoſſible.* Therefore, &c.

## PROP. VIII. Plate I. *Fig.* 15, *and* 16.

*If two triangles* ABC, DEF *have two ſides* AB, AC, *equal to two ſides* DE, DF, *each to each, and the baſe* BC *equal to the baſe* EF, *then the angles contained under the equal right-lines ſhall be equal,* viz. A *to* D.

Becauſe, BC (a) =EF, if the baſe BC be laid on the a *hyp.*
baſe EF, (b) they will agree: therefore whereas AB (c)= b *ax.* 8.
DE, and AC=DF, the point A will fall on D (for it cannot c *hyp.*
fall on any other point, by the precedent propoſition)
and ſo the ſides of the angles A and D are coincident;
(d) wherefore thoſe angles are equal. *Which was to be demonſtrated.* d 8. *ax.*

### *Coroll.*

1. Hence, triangles mutually equilateral are alſo mutually (x) equiangular.

2. Triangles mutually equilateral (x) are equal one to x 4. 1.
the other.

## PROP. IX. *Fig.* 19.

*To biſect, or divide into two equal parts, a right-lined angle given* BAC.

(a) Take AD=to AE, and draw the line DE; upon a 3. 1.
which (b) make an equilateral triangle DFE, draw the b 1. 1.
right-line AF; it ſhall biſect the Angle.

For AD (c) =AE, and the ſide AF is common, and the c *conſtr.*
baſe DF (c)=FE. (d) therefore the angle DAF=EAF. d 8. 1.
*Which was to be done.*

### *Coroll.*

Hence it appears how an angle may be cut into 4, 8, 16, 32, &c. equal parts, to wit, by biſecting each part again.

The method of cutting angles into any number of equal parts required, by a Rule and Compaſs, is as yet unknown to Geometricians.

## PROP. X. Plate I. *Fig.* 20.

*To biſect a right-line given* AB.

a 1. 1. Upon the line given AB (*a*) erect an equilateral triangle
b 9. 1. ABC; and (*b*) biſect the angle C with the right-line
CD. That line ſhall alſo biſect the line given AB.

c *conſtr.* For AC (*c*) =BC, and the ſide CD is common, and
d 4. 1. the angle ACD (*c*)=BCD. therefore AD (*d*)=BD.
*Which was to be done.*

The practice of this and the preceding propoſition is eaſily ſhewn by the Conſtruction of the 1ſt propoſition of this Book.

## PROP. XI. *Fig.* 20.

*From a point* D *in a right-line given* AB *to erect a right-line* DC *at right-angles.*

a 3. 1. (*a*) Take on either ſide of the point given DA=DB,
b 1. 1. upon the right-line AB (*b*) erect an equilateral triangle,
draw the line CD, and it will be the perpendicular required.

c *conſtr.* For the triangles DAC, DCB are mutually (*c*) equilateral,
d 8. 1. (*d*) therefore the angle ADC=CDB; (*e*) therefore
e 10. *def.* DC is perpendicular. *Which was to be done.*

The practice of this and the following is eaſily performed by the help of a ſquare.

## PROP. XII. *Fig.* 20.

*Upon an infinite right-line given* AB, *from a point given that is not in it, to let fall a perpendicular right-line* CD.

a 3 *poſt.* From the center C (*a*) deſcribe a circle cutting the
b 10. 1. right-line given in the points A and B. Then (*b*) biſect
AB in D, and draw the right-line in CD, which will be the perpendicular required.

Let the lines CA, CB be drawn. The triangles ADC
c *conſtr.* CDB are mutually (*c*) equilateral; (*d*) therefore the angles
d 8. 1. ADC, BDC are equal, and by (*e*) conſequence
e 10. *def.* right. (*e*) Wherefore DC is a perpendicular. *Which was to be done.*

## PROP. XIII. *Fig.* 1.

*When a right-line* DG *ſtanding upon a right-line* AB *maketh angles* DGA, DGB; *it maketh either two right-angles, or two angles equal to two right-angles.*

If the angles DGA, DGB be equal, (*a*) then they a *def.* 10.
make two right-angles; if unequal, then from the point
G (*b*) let there be erected a perpendicular GC. Becauſe b 11. 1.
the angle DGB (*c*) = to a right-angle+CGD, and the c 19. *ax.*
angle DGA (*d*) = to a right-angle—DGC; therefore the d 3. *ax.*
angle DGB+DGA (*e*) = to two right-angles. *Which* e 2. *ax.*
*was to be demonſtrated.*

### *Corollaries.*

1. Hence, if one angle DGB be right, the other DGA is alſo right; if one acute, the other is obtuſe, and ſo on the contrary.

2. If more right-lines than one ſtand upon the ſame right-line at the ſame point, the angles ſhall be equal to two right.

3. Two right-lines cutting each other make angles equal to four right-angles.

4. All the angles made about one point make four right-angles; as appears by *Coroll.* 2.

## PROP. XIV. Plate I. *Fig.* 1.

*If to any right-line* CG, *and a point therein* G *two right-lines, not drawn from the ſame ſide, do make the angles* CGA, CGB, *on each ſide equal to two right, the lines,* AG, GB, *ſhall make one ſtrait-line.*

If you deny it, let AG, GF make one right-line; then a 13. 1.
ſhall the angle CGA+CGF (*a*) = two right angles (*b*) = b *hyp.*
CGA+CGB. *Which is* (*c*) *abſurd.* c 9. *ax.*

## PROP. XV. *Fig.* 21.

*If two right-lines* AB, CD, *cut thro' one another, then are the two angles which are oppoſite,* viz. CEB, AED, *equal one to the other.*

For the angle AEC + CEB (*a*) = to two right right- a 13. 1.
angles = AEC + AED; (*b*) therefore CEB = AED. b 3. *ax.*
*Which was to be demonſtrated.*

### *Schol.* 1.

If to any right-line AB, and in it a point E, two right lines being drawn CE, ED, and not taken on the ſame ſide, make the vertical (or oppoſite) angles CEA, and BED equal, thoſe right-lines CE, ED, do meet directly and make one ſtrait line.

a 13. 1. For two right-angles are (*a*) equal to the angle CEA +
b 2. *ax.* CEB, (*b*) = CEB + BED. (*c*) Therefore CE, ED, are
c 14. .1 in a ſtrait line. *Which was to be demonſtrated.*

### *Schol.* 2. Plate I. *Fig.* 21.

If four right-lines EA, EB, EC, ED, proceeding from one point E, make the angles, vertically oppoſite, equal the one to the other, each two lines, AE, EB, and CE, ED, are placed in one ſtrait line.

For becauſe the angle AEC + AED + CEB + DEB
a 4. c 13. 1. (*a*) = to four right-angles, therefore the angle AEC +
b *hyp.* & AED (*b*) = CEB + DEB = to two right-angles. (*c*)
c2. *ax.* Therefore CED and AEB are ſtrait-lines. *Which was*
c 14. 1. *to be demonſtrated.*

### P R O P. XVI. *Fig.* 22.

*One ſide,* BC, *of any triangle* ABC *being produced, the outward angle* ACD *will be greater than either of the inward and oppoſite angles,* CAB, CBA.

a 10. 1. & Let the right-lines AH, BE, (*a*) biſect the ſides AC,
1. *poſt.* BC; from which lines produc'd, take (*b*) EF=BE,
b 3. 1. and HI, (*b*) = AH, and join FC, and IC; and produce ACG.

c *conſtr.* Becauſe CE (*c*) = EA, and EF (*c*) = EB, and the
d 15. 1. angle FEC (*d*) = BEA, the angle ECF (*e*) ſhall be equal
e 4. 1. to EAB. By the like argument is the angle ICH =
f 15. 1. ABH. Therefore the whole angle ACD (*f*) (BCG,)
g 9. *ax.* (*g*) is greater than either the angle CAB or ABC. *Which was to be demonſtrated.*

### P R O P. XVII. *Fig.* 23.

*Two angles of any triangle* ABC, *which way ſoever they are taken, are leſs than two right-angles.*

Let the ſide BC be produced. Becauſe the angle
a 13. 1. ACD + ACB (*a*) = two right-angles, and the angle
b 16. 1. ACD (*b*) ⊏ A, (*c*) therefore A + ACB ⊐ than two
c 4. *ax.* right-angles. After the ſame manner is the angle B + ACB ⊐ than two right-angles. Laſtly, the ſide AB being produced, the angle A + B will be alſo leſs than two right-angles. *Which was to be demonſtrated.*

*Coroll.*

1. Hence it follows that in every triangle wherein one angle is either right or obtuſe, the two others are acute angles.

2. If a right-line AE make unequal angles with another right-line BC, one acute AEB, the other obtuſe AEC, a perpendicular AB, let fall from any point A to the other line BC, ſhall fall on that ſide the acute angle is of.

For if AC, drawn on the ſide of the obtuſe-angle, be a perpendicular, then in the triangle AEC, ſhall the angles AEC + ACE be greater than two right-angles.
* *Which is contrary to the precedent prop.* * 17. 1.

3. All the angles of an equilateral triangle, and the two angles of an Iſoſceles triangle that are upon the baſe, are acute.

## PROP. XVIII. Plate I. *Fig.* 23.

*The greateſt ſide* AC *of every triangle* ABC *ſubtends the greateſt angle* ABC.

From AC (*a*) take away AF = AB, and join BF. a 3. 1.
(*b*) Therefore is the angle AFB = ABF. But b 5 1.
AFB (*c*) $\sqsubset$ C; therefore is ABF $\sqsubset$ C; (*d*) there- c 16. 1.
fore the whole angle ABC $\sqsubset$ C. After the ſame manſhall be ABC $\sqsubset$ A. *Which was to be demonſtrated.* d 9. *ax.*

## PROP. XIX. *Fig.* 23.

*In every triangle* ABC, *under the greateſt angle* B *is ſubtended the greateſt ſide* AC.

For if AB be ſuppoſed equal to AC, then will
be the angle B (*a*) = C, which is contrary to the a 5. 1.
Hypotheſis: And if AB $\sqsubset$ AC, then ſhall be the angle
C (*b*) $\sqsubset$ B, which is againſt the Hypotheſis. Where- b 18. 1.
fore rather AC $\sqsubset$ AB; and after the ſame manner AC $\sqsubset$ BC. *Which was to be demonſtrated.*

## PROP. XX. *Fig.* 24.

*Of every triangle* ABC, *two ſides* BA, AC, *any way taken, are greater than the ſide that remains* BC.

Pro-

a 3. 1. Produce the line BA, (*a*) and take AD=AC, and draw
b 5. 1. the line DC; (*b*) then ſhall the angle D be equal to
c 9. *ax.* ACD; (*c*) therefore is the whole angle BCD $\sqsubset$ D; (*d*)
d 19. 1. therefore BD (*e*) (BA+AC) $\sqsubset$ BC. *Which was to be*
e *conſtr.* *demonſtrated.*
& 2. *ax.*

## PROP. XXI. Plate I. *Fig.* 25.

*If from the utmoſt points of one ſide* BC, *of a triangle* ABC, *two right-lines* BD, CD, *be drawn to any point within the triangle, then are both thoſe two lines ſhorter than the two other ſides of the triangle* BA, CA; *but contain a greater angle,* BDC.

a 20. 1. Let BD be produced to E. Then is CE+ED (*a*) $\sqsubset$
b 4. *ax.* CD, and BD common to both, (*b*) then ſhall be DB+
DE+EC $\sqsubset$ CD+BD. Again, BA+AE (*a*) $\sqsubset$ BE;
(*b*) therefore BA+AC $\sqsubset$ BE+EC. Wherefore 1. BA+
c 16. 1. AC $\sqsubset$ BD+DC. 2. The angle BDC (*c*) $\sqsubset$ DEC (*c*) $\sqsubset$
A. Therefore the angle BDC $\sqsubset$ A. *Which was to be*
*demonſtrated.*

## PROP. XXII. *Fig.* 26.

*To make a triangle* FKG *of three right-lines* FK, FG, GK, *which ſhall be equal to three right-lines given* A,B,C. *Of which it is neceſſary that any two taken together be longer than the third.*

a 3. 1. From the infinite line DE (*a*) take DF, FG, GH, equal
b 3. *poſt.* to the lines given A,B,C. Then if from the (*b*) centers
F and G at the diſtances of FD and GH, two circles be
drawn cutting each other in K, and the right-lines KF,
c 15. *def.* KG be joined, the triangle FKG ſhall be made, (*c*) whoſe
ſides FK, FG, GK, are equal to the three lines DF, FG,
d 1. *ax.* GH, (*d*) that is, to the three lines given A, B, C. *Which*
*was to be done.*

## PROP. XXIII. *Fig.* 27, and 28.

*At a point* A *in a right-line given* AH, *to make a right-lined angle* A *equal to a right-lined angle given* D.

a 1. *poſt.* (*a*) Draw the right-line CF cutting the ſides of the
b 3. 1. angle given any ways; (*b*) make AG=CD; upon AG
c 22. 1. (*c*) raiſe a triangle equilateral to the former CDF, ſo
that AH be equal to DF, and GH to CF. then ſhall
d 8. 1. you have the angle A (*d*) =D. *Which was to be done.*

## PROP. XXIV. Plate I. *Fig.* 29, 30.

*If two triangles* ABC, DEF *have two ſides of the one triangle* AB, AC, *equal to two ſides of the other triangle* DE, DF, *each to each, and have the angle* A *greater than the angle* EDF *contained under the equal right-lines, they ſhall have alſo the Baſe* BC *greater than the baſe* EF.

(*a*) Let the angle EDG be made equal to A, and a 23. 1.
the ſide DG (*b*)$=$DF (*c*)$=$AC; and let EG, and FG b 3. 1.
be joined. c *hyp.*
1. *Caſe.* If EG falls above EF; Becauſe AB (*d*)$=$DE, d *hyp.*
and AC (*e*)$=$DG, and the angle A (*e*)$=$EDG; (*f*) e *conſtr.*
therefore is BC$=$EG. But becauſe DF (*e*)$=$DG; (*g*) f 4. 1.
therefore is the angle DFG$=$DGF; (*h*) therefore is the g 5. 1.
angle DFG$\sqsubset$EGF, and by conſequence the angle h 9. *ax.*
EFG, (*h*) $\sqsubset$EGF; (*k*) wherefore EG (BC) $\sqsubset$EF. k 19. 1.
2. *Caſe.* If the baſe EF coincides with the baſe EG,
(*l*) it is evident that EG (BC) $\sqsubset$ EF. *Fig.* 29, 31. l 9. *ax.*
3. *Caſe.* If EG falls below EF, then becauſe DG$+$
GE (*m*) $\sqsubset$ DF$+$FE, if from both be taken away DG, m 21. 1.
DF which are equal; EG (BC) remains (*n*) $\sqsubset$ EF. n 5. *ax.*
*Which was to be demonſtrated. Fig.* 29, 32.

## PROP. XXV. *Fig.* 29, 30.

*If two triangles* ABC, DEF, *have two ſides* AB, AC, *equal to two ſides* DE, DF, *each to each, and have the baſe* BC *greater than the baſe* EF, *they ſhall alſo have the angle* A *contained under the equal right-lines greater than the angle* EDF.

For if the angle A be ſaid to be equal to EDF, (*a*) a 4. 1.
then is the baſe BC$=$EF, which is againſt the Hypotheſis.
If it be ſaid the angle A $\sqsupset$ EDF, then (*b*) b 24. 1.
will be BC$\sqsupset$EF, which is alſo againſt the Hypotheſis.
Therefore A$\sqsubset$EDF. *Which was to be demonſtrated.*

## PROP. XXVI. *Fig.* 29, 31.

*If two triangles* BAC, EDG, *have two angles of the one* B, C, *equal to two angles of the other* E, DGE, *each to his correſpondent angle, and have alſo one ſide of the one equal to one ſide of the other, either that ſide which lyeth betwixt the equal angles, or that which is ſubtended under one of the equal angles; the other ſides alſo of the one ſhall be equal to* *the other ſides of the other, each to his correſpondent ſide, and the other angle of the one, ſhall be equal to the other angle of the other.*

1. *Hypotheſis.* Let B C be equal to E G, which are the ſides that lie between the equal angles, Then I ſay BA = ED, and AC=DG, and the angle A=EDG.
a 3. 1. For if it be ſaid that ED ⊏ BA, then (*a*) let EH be made equal to BA, and let the line GH be drawn.
b *conſtr.* Becauſe AB (*b*) = HE, and BC (*c*) = EG, and the an-
c *hyp.* gle B (*c*) = E, therefore ſhall be the angle EGH (*d*) =
d 4. 1. C (*e*) = DGE. (*f*) *Which is abſurd,* therefore AB = ED.
e *hyp.* After the ſame manner AC may be proved equal to DG,
f 9. *ax.* (*d*) then will the angle A be equal to EDG.

2. *Hyp.* Let A B be equal to D E, then I ſay B C = EG, and AC = DG, and the angle A=EDG. For if E G be greater than B C make EF = B C, and join
g *hyp.* DF. Now becauſe A B (*g*) = DE, and B C (*h*) = EF,
h *conſtr.* and the angle B (*g*) = E; therefore will be the angle
k 4. 1. EFD (*k*) = C (*l*) = EGD. (*m*) *Which is abſurd.* There-
l *hyp.* fore is BC = EG, and ſo as before, AC = DG, and
m 16. 1. the angle A = EDG. *Which was to be dem.*

## PROP. XXVII. Plate I. *Fig.* 33.

*If a right-line* EF; *falling upon two right-lines* AB, CD, *makes the alternate angles* A*b*F, D*c*E, *equal the one to the other, then are the right-lines* AB, CD, *parallel.*

If AB, CD be ſaid not to be parallel, produce them till they meet in O, which being ſuppoſed, the outward
a 16. 1. angle A*b*F will be (*a*) greater than the inward angle D*c*E, to which it was equal by Hypotheſis. *Which things are repugnant.*

## PROP. XXVIII. *Fig.* 33.

*If a right-line* E F, *falling upon two right-lines,* AB, CD, *makes the outward angle* A G E *of the one line equal to* C H G *the inward and oppoſite angle of the other on the ſame ſide, or make the inward angles on the ſame ſide,* A*b*H, C*c*G, *equal to two right-angles, then are the right-lines,* AB, CD, *parallel.*

*Hyp.* 1. Becauſe by Hypotheſis the angle A*b*E
a 15. 1. = C*c*G, (*a*) therefore are B*b*H, C*c*G, the alternate
b. 27. 1. angles equal; And (*b*) therefore are AB and CD parallel.

*Hyp.* 2. Becauſe by Hypotheſis the angle AGH+ a 13. 1.
CHG=to two right-angles,(*a*)=AGH+BGH,(*b*) there- b 3. *ax.*
fore ſhall the angle CHG=BGH ; and (*c*) therefore AB, c 27. 1.
CD, are parallel, *Which was to be demonſtrated.*

### PROP. XXIX. Plate I. *Fig.* 33.

*If a right-line* EF *falls upon two parallels,* AB, CD, *it will make both the alternate angles* DHG, AGH, *equal each to the other, and the outward angle* BGE *equal to the inward and oppoſite angle on the ſame ſide* DHG, *as alſo the inward angles on the ſame ſide* AGH, CHG, *equal to two right-angles.*

It is evident, that AGH+CHG=two right-angles ;
(*a*) otherwiſe AB, CD, would not be parallel, which is a 13. *ax.*
contrary to the Hypotheſis : But moreover the angle b 13. 1.
DHG+CHG (*b*)=two right-angles ; therefore is DHG c 3 *ax.*
(*c*) = AGH (*d*) =BGE. *Which was to be dem.* d 15. 1.

### *Coroll. Fig.* 7.

Hence it follows that every parallelogram AD having one angle right A, the reſt are alſo right.

For A+B (*a*)=two right-angles. Therefore, where- a 29. 1.
as A is right, (*b*) B muſt be alſo right. By the ſame b 3. *ax.*
argument are C and D right-angles.

### PROP. XXX. *Fig.* 21.

*Right-lines* (AB, MN) *parallel to one and the ſame right-line* KL, *are alſo parallel the one to the other.*

Let CD cut the three right-lines given any ways.
Then becauſe AB, MN are parallel, the angle BEI
will be (*a*) =LHI. Alſo becauſe KL and MN are a 29. 1.
parallel, the angle LHI will be (*a*)= MIC (*b*) There- a 29. 1.
fore the angle BEI=MIC ; (*c*) whence AB and MN b 1. *ax.*
are parallel. *Which was to be demonſtrated.* c 27. 1.

### PROP, XXXI. *Fig.* 34.

*From a point given* A, *to draw a right-line* AE, *parallel to a right-line given* BC.

From the point A draw a right-line AD to any point
of the given right-line ; with which at the point thereof
(*a*) A make an angle DAE =ADC ; (*b*) then will AE a 23. 1.
and BC be parallel. *Which was to be done.* b 27. 1.

## PROP. XXXII. Plate I. *Fig.* 35.

*Of any triangle* ABC *one ſide* BC *being drawn out, the outward angle* ACD *ſhall be equal to the two inward oppoſite angles* A, B, *and the three inward angles of the triangle,* A, B, ACB, *ſhall be equal to two right-angles.*

a 31. 1. From C *(a)* draw CE parallel to BA. Then is the
b 29. 1. angle A *(b)* = ACE, and the angle B *(b)* = ECD.
c 2 *ax.* Therefore A+B *(c)* = ACE+ECD *(d)* = ACD. *Which*
d 19. *ax.* *was to be demonſtrated.*
e 13. 1. I affirm ACD+ACB *(e)* = two right-angles; *(f)*
f 1. *ax.* therefore A+B+ACB = two right-angles. *Which was to be demonſtrated.*

### *Coroll.*

1. The three angles of any triangle taken together are equal to the three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle, two angles (taken ſeverally, or together) be equal to two angles of another triangle (taken ſeverally, or together) then is the remaining angle of the one equal to the remaining angle of the other. In like manner, if two triangles have one angle of the one equal to one of the other, then is the ſum of the remaining angles of the one triangle equal to the ſum of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right-angle. Likewiſe, that angle in a triangle which is equal to the other two, is it ſelf a right-angle.

4. When in an Iſoſceles the angle made by the equal ſides is right, the other two upon the baſe are each of them half a right-angle.

5. An angle of an equilateral triangle makes two third parts of a right-angle. For one third of two right-angles is equal to two thirds of one.

### *Schol.*

By the help of this propoſition you may know how many right-angles the inward and outward angles of a right-lined figure make; as may appear by theſe two following Theorems.

## *THEOREM* I.

*All the angles of a right-lined figure do together make twice as many right-angles, abating four, as there are ſides of the figure.*

From any point within the figure let right-lines be drawn to all the angles of the figure, which ſhall reſolve the figure into as many triangles as there are ſides of the figure. Wherefore, whereas every triangle affords two right-angles, all the triangles taken together will make up twice as many right-angles as there are ſides. But the angles about the ſaid point within the figure make up four right; therefore, if from the angles of all the triangles you take away the angles which are about the ſaid point, the remaining angles, which make up the angles of the figure, will make twice as many right-angles, abating four, as there are ſides of the figure. *Which was to be demonſtrated.*

### *Coroll.*

Hence all right-lined figures of the ſame ſpecies have the ſums of their angles equal.

## *THEOREM* II.

*All the outward angles of any right-lined figure, taken together, make up four right-angles.*

For every inward angle of a figure, with the outward angle of the ſame, make two right-angles; therefore all the inward angles, together with all the outward, make twice as many right-angles as there are ſides of the figure; but (as has been juſt ſhewn) all the inward angles, with four right, make twice as many right as there as ſides of the figure, therefore the outward angles are equal to four right-angles. *Which was to be demonſtrated.*

### *Coroll.*

All right-lined figures, of whatſoever ſpecies, have the ſums of their outward angles equal.

## PROP. XXXIII. Plate I. *Fig.* 7.

*If two equal and parallel lines* AB, CD, *be joyned together with two other right-lines,* AC, BD, *then are thoſe lines alſo equal and parallel.*

Draw a line from C to B. Now becauſe AB and CD
a 29. 1. are parallel, and the angle ABC (*a*) = BCD; and alſo
by hypotheſis AB = CD, and the ſide CB common,
b 4. 1. therefore is AC (*b*) = BD, and the angle ACB (*b*) =
c 27. 1. DBC (*c*) whence alſo AC, BD, are parallel,

## PROP. XXXIV. *Fig.* 7.

*In parallelograms, as* ABDC, *the oppoſite ſides* AB, CD, *and* AC, BD, *are equal each to the other; and the oppoſite angles* A, D, *and* ABD, ACD, *are alſo equal; and the diameter* BC *biſects the ſame.*

a *hyp.* Becauſe AB, CD, (*a*) are parallel, (*b*) therefore is the
b 29. 1. angle ABC = BCD. Alſo becauſe AC, BD, are (*a*) parallel, (*b*) therefore is the angle ACB = CBD; (*c*) there-
c 2. *ax.* fore the whole angle ACD = ABD. After the ſame
manner is A = D. Moreover becauſe the angles ABC,
ACB, lie at each end of the ſide CB, and are equal to
d 26. 1. BCD, CBD, (*d*) therefore is AC = BD, and AB (*d*) =
CD, and ſo the triangle ABC = CBD. *Which was to be demonſtrated.*

### *Schol.*

*Every four-ſided figure* ABDC, *having the oppoſite ſides equal, is a parallelogram.*

a 27. 1. For by 8. 1. the angle ABC = BCD; (*a*) wherefore
AB, CD, are parallel. In like manner is the angle BCA
b 35. *def.* 1. = CBD; (*a*) wherefore AC, BD, are alſo parallel. (*b*)
Therefore ABCD is a parallelogram. *Which was to be demonſtrated.*

From hence we may more expeditiouſly draw a
*Fig.* 36. parallel CG to a right-line given, AB, thro' a point
aſſigned, C.

Take in the line AB any point, as E. From the centers E and C at any diſtance draw two equal circles EF, CD. From the center F with the diſtance EC draw a circle FD, which ſhall cut the former circle CD in the point D. Then ſhall the line drawn CG be parallel to AB, for, as it was before demonſtrated, CEFD is a parallelogram.

## PROP. XXXV. Plate I. *Fig.* 37.

*Parallelograms,* BCDA, BCFE, *which ſtand upon the ſame baſe* BC, *and between the ſame parallels* AF, BC, *are equal one to the other.*

For AD (*a*) = BC (*a*) = EF, add DE common to a 34. 1.
both; (*b*) then is AE = DF. But alſo AB (*a*) = b 2. *ax.*
DC, and the angle A (*c*) = CDF. (*d*) Therefore is c 29. 1.
the triangle ABE=DCF. Take away D *g* E common d 4. 1.
to both triangles ; (*e*) then is the Trapezium AB *g* D = e 3. *ax.*
E *g* CF ; add B *g* C, common to both ; (*f*) then is the f 2. *ax.*
parallelogram ABCD = EBCF. *Which was to be demonſtrated.*

The demonſtration of any other caſes, is not unlike, but much more plain and eaſy.

### *Schol. Fig.* 7.

If the ſide AB, of a right-angled parallelogram ABCD, be conceived to be carried along perpendicularly thro' the whole line AC, or AC thro' the whole line AB, the area or content of the rectangle ABCD ſhall be produc'd by that motion. Hence a rectangle is ſaid to be made by the drawing or multiplication of two contiguous ſides. For example; let AB be ſuppoſed four foot, and AC three; draw three into four, there will be produced twelve ſquare feet for the area of the rectangle.

This being ſuppoſed, the dimenſion of any parallelogram (EBCF) is found out by this theorem. For the area thereof is produced from the altitude BA drawn into the baſe BC. For the area of the rectangle AC = parallelogram EBCF, is made by the drawing of BA into BC, therefore, &c. *Fig.* 37.

## PROP. XXXVI. *Fig* 37.

*Parallelograms* BCDA, GHFE, *ſtanding upon equal baſes* BC, GH, *and betwixt the ſame parallels* AF, BH, *are equal one to the other.*

Draw BE and CF. Becauſe BC (*a*) = GH (*b*) = a *hyp.*
EF, (*c*) therefore is BCFE a parallelogram. Whence b 34. 1.
the parallelogram BCDA (*d*) = BCFE (*d*) = GHFE. c 33. 1.
*Which was to be demonſtrated.* d 35. 1.

### PROP. XXXVII. Plate I. *Fig.* 38.

*Triangles,* BCA, BCD, *ſtanding upon the ſame baſe* BC, *and between the ſame parallels* BC, EF, *are equal the one to the other.*

(*a*) Draw BE parallel to CA, (*a*) and CF parallel to BD. Then is the triangle BCA (*b*) = half Pgr. BCAE (*c*) = half BDFC (*b*) = BCD. *Which was to be demonſtrated.*

a 31. 1.
b 34. 1.
c 35. 1.
*and* 7. *ax.*

### PROP. XXXVIII. *Fig.* 38.

*Triangles,* BCA, DFC, *ſet upon equal baſes* BC, DF, *and between the ſame parallels* EF, BC, *are equal the one to the other.*

Draw EB parallel to AC, and DB parallel to FC. Then is the triangle BCA (*a*) = half Pgr. BCAE (*b*) = half BCDF (*c*) = CDF. *Which was to be demonſtrated.*

a 34. 1.
b 36. 1.
*and* 7. *ax.*
c 34. 1.

*Schol.*

If the baſe BC be greater than DF, then is the triangle BAC ⊏ DFC, and ſo on the contrary.

### PROP. XXXIX. *Fig.* 39.

*Equal triangles* BCA, BCD, *ſtanding on the ſame baſe* BC, *and on the ſame ſide are alſo between the ſame parallels* AD, BC.

If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF (*a*) = CBA (*b*) = CBD. (*c*) *Which is abſurd.*

a 37. 1.
b *hyp.*
c 9. *ax.*

### PROP. XL. *Fig.* 40.

*Equal triangles* BCA, CFD, *ſtanding upon equal baſes* BC, CF, *and on the ſame ſide, are betwixt the ſame parallels.*

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle CFH (*a*) = BCA (*b*) = CFD. (*c*) *Which is abſurd.*

a 38. 1.
b *hyp.*
c 9. *ax.*

### PROP. XLI. *Fig.* 38.

*If a Pgr.* AEBC *have the ſame baſe* BC *with the triangle* BCE, *and be between the ſame parallels* DF, BC, *then is the Pgr.* AEBC *double to the triangle* BCE.

Let the line AC be drawn. Then is the triangle BCA a 37. 1.
(*a*) =BCE; therefore is the Pgr. ABCD (*b*) =2BCA b 34.1.
(*c*)= 2BCE. *Which was to be demonſtrated.* c 6. *ax.*

*Schol.*

From hence may the area of any triangle BCE be found, for whereas the area of the Pgr. ABCD is produced by the altitude drawn into the baſe, therefore ſhall the area of a triangle be produced by half the altitude drawn into the baſe, or half the baſe drawn into the altitude; thus, if the baſe BC be 8, and the altitude 7, then is the area of the triangle BCE 28.

## PROP. XLII. Plate I. *Fig.* 41.

*To make a Pgr.* ECGF *equal to a triangle given* ABC *in an angle equal to a right-lined angle given.*

Through A (*a*) draw AG parallel to BC, (*b*) make the a 31. 1.
angle BCG=the angle given; (*c*) biſect the baſe BC in b 23. 1.
E, and draw EF parallel to CG; then is the problem c 10. 1.
reſolved.

For (AE being drawn) the angle ECG is equal to the d 38. 1.
given angle by conſtruction, and the triangle BAC (*d*) e 41. 1.
=2 AEC (*e*)= Pgr. ECGF. *Which was to be done.*

## PROP. XLIII. *Fig.* 9.

*In every Pgr.* AGEL, *the complements* LD, GD, *of thoſe Pgrs.* CK, KF, *which ſtand about the diameter, are equal one to the other.*

For the triangle AEL (*a*)=AEG, and the triangle
ADC (*a*) =ADB; and the triangle DEK (*a*) =DEF (*b*) a 34. 1.
Therefore the Pgr. LD=DG. *Which was to be demon-* b 3 *ax.*
*ſtrated.*

## PROP. XLIV. *Fig.* 42.

*To a given right-line* A, *to apply a parallelogram* FL, *equal to a given triangle* LBC, *in a given angle* C.

(*a*) Make a Pgr. FD equal to the triangle LBC, ſo that a 42. 1.
the angle GFE may be equal to C. Produce GF till
FH be equal to the line given A. Through H (*b*) draw b 31. 1.
IL parallel to EF, which let DE produced meet in I,
let DG produced meet with a right-line drawn from I
through F in the point K; thro' K (*b*) draw KL parallel
to GH, which let EF continued meet at M, and IH
at L. Then ſhall FL be the Pgr. required.

c [illegible]. 1. For the Pgr. FL (*c*) =FD= LBC; (*d*) and the angle
d 15. 1. MFH=GFE=C. *Which was to be done.*

## PROP. XLV. Plate I. *Fig.* 43, and 44.

*Upon a right-line given* FG, *and in a given angle* A, *to make a Pgr.* FL, *equal to a right-lined figure given* ABCD.

Reſolve the right-lined figure given into two triangles
a 44. 1. BAD, BCD, then (*a*) make a Pgr. FH=BAD, ſo that
the angle F may be equal to A. FI being produced,
b 19. *ax.* (*a*) make on HI the Pgr. IL=BCD. Then is the Pgr.
c *conſtr.* FL (*b*)=FH+IL (*c*)=ABCD. *Which was to be done.*

### *Schol.*

Hence is eaſily found the exceſs, whereby any right-lined figure exceeds a leſs right-lined figure.

## PROP. XLVI. *Fig.* 6.

*Upon a right-line given* CD *to deſcribe a ſquare* DB.
a 11. 1. (*a*) Erect two perpendiculars CB, DA, (*b*) equal to the
b 3. 1. line given CD; then join BA, and the thing reqired is
done.
c *conſtr.* For, whereas the Angle C+D (*c*) = two right-an-
d 28. 1. gles (*d*) therefore are AD, BC parallel. But they are
e *conſtr.* alſo (*e*) equal; (*f*) therefore AB, DC are both parallel
f 33. 1. and equal; therefore the figure AC is a Pgr. and equila-
g *cor.* 29. 1. teral. Moreover the angles are all right, (*g*) becauſe one
h 29. *dif.* A, is right; (*h*) therefore DB is a ſquare. *Which was
to be done.*

After the ſame manner you may eaſily deſcribe a rectangle contained under two right-lines given.

## PROP. XLVII. *Fig.* 45.

*In right-angled triangles* BAC, *the ſquare* BE, *which is made on the ſide* BC *that ſubtends the right-angle* BAC, *is equal to both the ſquares* BG, CH, *which are made on the ſides* AB, AC, *containing the right-angle.*

Join AE, and AD; and draw AM parallel to CE.
a 12. *ax.* Becauſe the angle DBC (*a*) =FBA, add the augle ABC
common to them both; then is the angle ABD= FBC.
b 29. *def.* Moreover, AB (*b*) =FB, and BD (*b*) =BC; (*c*) there-
c 4. 1. fore is the triangle ABD=FBC. But the Pgr. BM (*d*)=
d 41. 1. 2 ABD,

$2\,ABD$, and the Pgr. (*d*) $BG = 2FBC$ (for GAC is one d 41. 1.
right-line, by Hypotheſis, and 14. 1.) (*e*) therefore is the e 6. *ax.*
Pgr. $BM = BG$. By the ſame way of argument is the
Pgr. $CM = CH$. Therefore is the whole $BE =$ (*f*) $BG +$ f 2. *ax.*
$CH$. *Which was to be demonſtrated.*

*Schol.*

This moſt excellent and uſeful theorem hath deſerved the title of *Pythagoras* his theorem, becauſe he was the inventor of it. By the help of which the addition and ſubtraction of ſquares are performed; to which purpoſe ſerve the two following problems.

## PROBLEM I. Plate I. *Fig.* 46.

*To make one ſquare equal to any number of ſquares given.* Andr.
Let three ſquares be given, whereof the ſides are Tacq.
AB, BC, CE. (*a*) Make the right-angle FBZ, having a 11. 1.
the ſides infinite; and on them transfer AB and BC;
join AC, then is $ACq$ (*b*) $= ABq + BCq$. Then trans- b 47. 1.
fer AC from B to X, and CE the third ſide given
from B to E; join EX. (*b*) Then is $EXq = EBq$
$(CEq) + BXq\ (ACq)$ (*c*) $= CEq + ABq + BCq$. c 2. *ax.*
*Which was to be demonſtrated.*

## PROBLEM II. *Fig.* 2.

*Two unequal right-lines being given* AE, EG, *to make a ſquare equal to the difference of the two ſquares of the given lines* AE, EG.

From the center E, at the diſtance of AE, deſcribe
a circle, and from the point G erect a perpendicular
FG, meeting the circumference in F; and draw EF.
(*a*) Then is $EFq\ (EAq) = EGq + FGq$, (*b*) There- a 47. 1.
$EAq - EGq = GFq$. *Which was to be done.* b 3. *ax.*

## PROBLEM III. *Fig.* 45.

*Any two ſides of a right-angled triangle* ABC, *being known, to find out the third.*

Let the ſides AB, AC, encompaſſing the right-angle,
be, the one 6 foot, the other 8. Therefore, whereas
$ACq + ABq = 64 + 36 = 100 = BCq$, thence is BC
$= \sqrt{100} = 10$. 47. 1.

Now, let the ſides AB, BC, be known, the one 6 foot, the other 10, Therefore ſince $BCq - ABq =$
47. 1. $100 - 36 = 64 = ACq$, whence $AC = \sqrt{64} = 8$.
*Which was to be done.*

## P R O P. XLVIII. Plate I. *Fig.* 47.

*If the ſquare made upon one ſide* BC *of a triangle be equal to the ſquares made on the other ſides of the triangle* AB, AC, *then the angle* BAC *comprehended under the two other ſides of the triangle* AB, AC, *is a right-angle.*

Perpendicular to AC draw $AD = AB$, and join CD.
a 47. 1. Now is (*a*) $CDq = ADq + ACq = ABq + ACq =$
* *See the* $BCq$. * Therefore is $CD = BC$. And therefore the
*following* triangles CAB, CAD, are mutually equilateral. Where-
*theor.* fore the angle CAB (*b*) = CAD (*c*) = a right-angle.
b 8. 1. *Which was to be demonſtrated.*
c *conſtr.*

### *Schol.*

We aſſumed in the demonſtration of the laſt Propoſition, $CD = BC$, becauſe CDq was equal to BCq: Our aſſumption we prove by the following

## *T H E O R E M. Fig.* 48.

*The ſquares* AF, CG *of equal right-lines* AB, CD, *are equal one to the other: And the ſides* AB, CD, *of equal ſquares* AF, CG, *are equal one to the other.*

1. *Hypotheſis.* Draw the diameters EB, HD. Then
a 34. 1. it is evident that AF is (*a*) equal to the triangle EAB
b 4. 1. & twice taken, and (*b*) equal to the triangle HCD twice
6. *ax.* taken, and equal to (*a*) CG. *Which was to be demonſtrated.*
a 46. 1. 2. *Hyp.* If it may be, let CD be greater than AB.
b 1. *part*, Make $CT = AB$, and let $CS = CTq$. Therefore is
c *hyp.* CS (*b*) = EB (*c*) = CG. *Which is abſurd.*
d 9. *ax.*

### *Corol.*

After the ſame manner any rectangles equilateral one to another, are demonſtrated to be alſo equal.

## *The End of the firſt* BOOK.

# The SECOND BOOK OF EUCLIDE's ELEMENTS.

## *Definitions.*

I. EVery right-angled Parallelogram ABCD, is ſaid to be contained under two right-lines AB, AC, comprehending a right-angle. Plate I. *Fig.* 7.

*Therefore when you meet with ſuch as theſe, the rectangle under* BA, AC, *or more briefly the rectangle* BAC, *or* BA × AC (or ZA, *for* Z × A) *the rectangle meant is that which is contained under the right-lines* BA, AC, *ſet at right-angles.*

II. In every Pgr. ALGE, any one of thoſe parallelograms which are about the diameter, together with its two complements is called a Gnomon. *As the Pgr.* FB + DL + FK (BEC) *is a Gnomon; and likewiſe the Pgr.* FB, DL + BC (FAK) *is a Gnomon.* *Fig.* 9.

### PROP. I. *Fig.* 49.

*If two right-lines* AF, AB, *are given, and one of them* AB *be divided into as many parts or ſegments as you pleaſe; the rectangle comprehended under the two whole right-lines* AB, AF, *ſhall be equal to all the rectangles contained under the whole line* AF, *and the ſeveral ſegments,* AD, DE, EB.

(*a*) Set AF perpendicular to AB. Thro' F (*a*) draw an infinite line FG perpendicular to AF. From the points D, E, B, erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is (*b*) equal to the rectangles AH, DI, EG, that is (becauſe DH, EI, AF, (*c*) are equal) to the rectangles under AF, AD, under AF, DE, under AF, EB. *Which was to be demonſtrated.*

a 11. 1.
b 19 *ax* 1.
c 34. 1.

### *Schol.*

*If two right-lines given are both divided into how many parts ſoever, one whole multiplied into the other ſhall bring out the ſame product, as the parts of one multiplied into the parts of the other.*

For let $Z$ be $= A + B + C$, and $Y = D + E$; then, a 1. 2. becauſe $DZ$ (*a*) $= DA + DB + DC$, and $EZ$ (*a*) $=$ b 2. *ax.* $EA + EB + EC$, and $YZ$ (*a*) $= DZ + EZ$, (*b*) Therefore $ZY$ will be $= DA + DB + DC + EA + EB + EC$. *Which was to be demonſtrated.*

*From hence we have a method of multiplying compound lines into compound ones. For if the rectangles of all the parts be taken, their ſum ſhall be equal to the rectangle of the wholes.*

But whenſoever in the multiplication of lines into themſelves you meet with theſe ſigns — intermingled with theſe +, you muſt alſo have particular regard to the ſigns. For of + multiplied into — ariſeth —; but of — into — ariſeth + *ex. gr.* let $+ A$ be multiplied into $B - C$; then becauſe $+ A$ is not affirmed of all B, but only of that part of it, whereby it exceeds C, therefore AC muſt remain denied; ſo that the product will be $AB - AC$. Or thus; becauſe B conſiſts of the parts C and * 1. 2. $B - C$, * thence $AB = AC + A \times B - C$, take away AC from both, then $AB - AC = A \times B - C$. In like manner, if $- A$ be to be multiplied into $B - C$, then ſince by virtue of the ſign —, A is not denied of all B, but only of ſo much as it exceeds C, therefore AC muſt remain affirmative, whence the product will be $- AB + AC$. Or thus; becauſe $AB = AC + A \times B - C$; take away all from both ſides, and there will be $- AB = - AC - A \times B - C$; add AC to both, and it will be $- AB + AC = - A \times B - C$.

This being ſufficiently underſtood, the nine following propoſitions, and innumerable others of that kind, ariſing from the comparing of lines multiplied into themſelves (which you may find done to your hand in *Vieta*, and other analytical Writers) are demonſtrated with great facility, by reducing the matter for the moſt part to almoſt a ſimple work.

Furthermore, * it appears that the product ariſing * 19. *ax.* from the multiplication of any magnitude into the parts of any number is equal to the product ariſing from the multiplication of the ſame into the whole number: As $5 A + 7 A = 12 A$, and $4 A \times 5 A + 4 A \times 7 A =$

$4A \times 12A$. Wherefore what is here delivered of the multiplying of right-lines into themſelves, the ſame may be underſtood of the multiplying of numbers into themſelves, ſo that whatſoever is affirmed concerning lines in the nine following Theorems, holds good alſo in numbers ; ſeeing they all immediately depend on, and are deriv'd from this firſt.

### PROP. II. Plate I. *Fig.* 50.

*If a right-line* Z *be divided any wiſe into two parts, the rectangles comprehended under the whole line* Z, *and each of the ſegments* A, E, *are equal to the ſquare made of the whole line* Z.

I ſay that $ZA + ZE = Zq$. For take $B = Z$ ; (*a*) a 1. 2.
then is $BA + BE = BZ$, that is (becauſe $B = Z$) $ZA + ZE = Zq$. *Which was to be demonſtrated.*

### PROP. III. *Fig.* 50.

*If a right-line* Z, *be divided any wiſe into two parts, the rectangle comprehended under the whole line* Z, *and one of the ſegments* E, *is equal to the rectangle made of the ſegments* A, E, *and the ſquare deſcribed on the ſaid ſegment* E. a 1. 2.
I ſay $ZE = AE + Eq$. (*a*) For $EZ = EA + Eq$.

### PROP. IV. *Fig.* 50.

*If a right-line* Z *be cut any wiſe into two parts, the ſquare deſcribed on the whole line* Z, *is equal to the ſquares deſcribed on the ſegments* A, E, *and to twice a rectangle made of the ſegments* A, E, *taken together.*

I ſay that $Zq = Aq + Eq + 2AE$. For $ZA$ (*a*) $= Aq$ a 3. 2.
$+ AE$, and $ZE$ (*a*) $= Eq + EA$. Therefore whereas b. 2. 2.
$ZA + ZE$ (*b*) $= Zq$, (*c*) thence is $Zq = Aq + Eq +$ c 1. *ax.*
$2AE$. *Which was to be demonſtrated.*

*Otherwiſe thus*; Upon the right-line AB make the *Fig.* 51.
ſquare AD, and draw the diameter EB ; thro' C, the point wherein the line AB is divided, draw the perpendicular CF ; and thro' the point G draw HI parallel to AB.

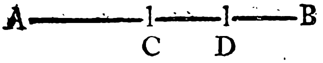
Becauſe the angle $EHG = A$ is a right-angle, and d 4. *cor.*
AEB is (*d*) half a right (*e*) therefore is the remaining 32. 1.
angle HGE half a right-angle. Therefore is HE (*f*) $=$ e 32. 1.
HG (*g*) $=$ EF (*g*) $=$ AC, ſo that HF (*h*) is the ſquare of f 6. 1.
the right-line AC. After the ſame manner is CE proved g 34. 1.
to h 29. *def.*

to be CBq. Therefore AG, GD, are rectangles under
k 19. *ax.* 1. AC, CB, wherefore the whole ſquare AD (*k*) $= ACq + CBq + 2ACB$. *Which was to be demonſtrated.*

*Coroll.*

1. Hence it appears that the Pgrs. which are about the diameter of a ſquare are alſo ſquares themſelves.
2. That the diameter of any ſquare biſects it angles.
3. That if $a = \frac{1}{2} Z$, then is $Zq = 4\,Aq$, and $Aq = \frac{1}{4} Zq$. As on the contrary, if $Zq = 4\,Aq$, then is $A = \frac{1}{2} Z$.

## PROP. V.



*If a right-line* AB *be cut into equal parts* AC, CB, *and into unequal parts* AD, DB, *the rectangle comprehending under the unequal parts* AD, DB, *together with the ſquare that is made of the difference of the parts* CD, *is equal to the ſquare deſcribed on the half line* CB.

I ſay that $CBq = ADB + CDq$.

a 4. 1.
b 3. 2.
c *hyp.*

For theſe are all equal.
$$\begin{cases} CBq. \\ (a)\ CDq + CDB + DBq + CDB. \\ CDq + (b)\ CBD\ (c)\ (AC \times BD) + CDB. \\ CDq + (d)\ ADB. \end{cases}$$

d 1. 2. This theorem is ſomewhat differently expreſs'd and more eaſily demonſtrated thus; *A rectangle made of the ſum and the difference of two right-lines* A, E, *is equal to the difference of the ſquares of thoſe lines.*

* Sch. 1. 2. For if $A + E$ be multiplied into $A - E$, * there ariſeth $Aq - AE + EA - Eq = Aq - Eq$. *Which was to be demonſtrated.*

*Schol.* Plate I. *Fig.* 52.

If the line AB be divided otherwiſe, (*viz.*) nearer to the point of biſection, in E; then is $AEB \sqsubset ADB$.

a 5. 2. & 3. *ax.* For AEB $(a) = CBq - CEq$, and ADB $(a) = CBq - CDq$. Therefore, whereas $CDq \sqsubset CEq$, thence is $AEB \sqsubset ADB$. *Which was to be demonſtrated.*

*Coroll.*

1. Hence is $ADq + DBq \sqsubset AEq + EBq$. For $ADq + DBq + 2\,ADB\ (b) = ABq\ (b) = AEq + EBq + 2AEB$. Therefore becauſe $2AEB \sqsubset 2ADB$, ſhall $ADq + DBq \sqsubset AEq + EBq$. *Which was to be demonſtrated.* b 4. 2.

c 3. *ax.* 2. Hence is $ADq + DBq - AEq\ (c) - EBq = 2AEB - 2\,ADB$.

PROP.

## PROP. VI. Plate I. *Fig.* 53.

*If a right-line* A *be divided into two equal parts, and another right-line* E, *added to the ſame directly in one right-line, then the rectangle comprehended under the whole and the line added,* (viz. A$+$E,) *and the line added* E, *together with the ſquare which is made of* $\frac{1}{2}$ *the line* A, *is equal to the ſquare of* $\frac{1}{2}$ A$+$E *taken as one line.*

I ſay that $\frac{1}{4}$ Aq. ( (*a*) Q. $\frac{1}{2}$ A) $+$ AE $+$ Eq $=$ Q. $\frac{1}{2}$ a 4 & 3.
A$+$E. (*a*) For, Q. $\frac{1}{2}$ A$+$E$=\frac{1}{4}$Aq$+$Eq$+$AE. *Which* *Cor.* 4. 2.
*was to be demonſtrated.*

*Coroll.*

Hence it follows, that if 3 right-lines E, E$+\frac{1}{2}$A, E$+$A be in arithmetical proportion, then the rectangle contained under the extreme terms E, E$+$A, together with the ſquare of the difference $\frac{1}{2}$ A, is equal to the ſquare of the middle term E$+\frac{1}{2}$A.

## PROP. VII. *Fig.* 50.

*If a right-line* Z *be divided any wiſe into two parts, the ſquare of the whole line* Z, *together with ſquare made of one of the ſegments* E, *is equal to a double rectangle comprehended under the whole line* Z, *and the ſaid ſegment* E, *together with the ſquare made of the other ſegment* A.

I ſay, that Zq$+$Eq$=$2ZE$+$Aq. For Zq (*a*)$=$Aq$+$ a 4. 2.
Eq$+$2AE, and 2 ZE (*b*)$=$2 Eq$+$2AE. *Which was to* b 3. 2.
*be demonſtrated.*

*Coroll.*

Hence it follows, that the ſquare of the difference of any two lines Z, E, is equal to the ſquare of both the lines leſs by a double rectangle comprehended under the ſaid lines.

(*c*) For Zq$+$Eq$-$2ZE$=$Aq$=$Q. Z$-$E. c 7. 2. *and*
3. *ax.*

## PROP. VIII. *Fig.* 50.

*If a right-line* Z *be divided any wiſe into two parts, the rectangle comprehended under the whole line* Z, *and one of the ſegments* E *four times, together with the ſquare of the other ſegment* A, *is equal to the ſquare of the whole line* Z, *and the ſegment* E, *taken as one line* Z$+$E.

I ſay, that 4ZE$+$Aq$=$Q. Z $+$E. For 2 ZE (*a*)$=$ a 7. 2. *and*
Zq $+$ Eq $-$ Aq. Therefore 4 ZE $+$ Aq $=$ Zq $+$ Eq 3. *ax.*
$+$ 2ZE (*b*)$=$Q. Z $+$ E. *Which was to be demonſtrated.* b 4. 2.

### PROP. IX. Plate I. *Fig.* 54.

*If a right-line* AB *be divided into equal parts* AC, CB *and into unequal parts* AD, DB, *then are the ſquares of the unequal parts* AD, DB, *together, double to the ſquare of the half line* AC, *and to the ſquare of the difference* CD.

I ſay that $ADq + DBq = 2\,ACq + 2CDq$. For ADq
a 4. 2. $+ DBq$ (*a*) $= ACq + CDq + 2\,ACD + DBq$. But
b *hyp.* $2ACD$ (*b*) $(2BCD) + DBq$ (*c*) $= CBq\ (ACq) + CDq$.
c 7. 2. (*d*) Therefore $ADq + DBq = 2\,ACq + 2CDq$. *Which*
d 2. *ax.* *was to be demonſtrated.*

This may be otherwiſe delivered and more eaſily demonſtrated thus; the aggregate of the ſquares made of the ſum and the difference of two right-lines A, E, is equal to the double of the ſquares made from thoſe lines.

a 4, 2. For Q. $A + E$ (*a*) $= Aq + Eq + 2AE$, and Q. $A -$
b *cor.* 7. 2. $E$ (*b*) $= Aq + Eq - 2AE$. Theſe added tegether make $2Aq$
$+ 2Eq$. *Which was to be demonſtrated.*

### PROP. X. *Fig.* 53.

*If a right-line* A *be divided into two equal parts, and another line be added in a right-line with the ſame, then is the ſquare of the whole line together with the added line (as being one line) together with the ſquare of the added line* E, *double to the ſquare of half* A, *and the added line* E, *taken as one line.*

a 4. 2. I ſay that $Eq +$ Q. $A + E$, *i. e.* (*a*) $Aq + 2\,Eq + 2AE$
b *cor.* 4. 2. $= 2$ Q. $\frac{1}{2} A + 2$ Q. $\frac{1}{2} A + E$, For 2 Q. $\frac{1}{2} A$ (*b*) $= \frac{1}{2} Aq$.
c 4. 2. And 2 Q. $\frac{1}{2} A + E$ (*c*) $= \frac{1}{2} Aq + 2\,Eq + 2AE$. *Which*
*was to be demonſtrated,*

### PROP. XI. *Fig.* 55.

*To cut a right-line given* AB, *in a point* G, *ſo that the rectangle comprehended under the whole line* AB, *and one of the ſegments* BG, *ſhall be equal to the ſquare, that is made of the other ſegment* AG.

a 46. 1. Upon AB (*a*) deſcribe the ſquare AC (*b*) Biſect the ſide
b 10. 1. AD in E, and draw the line EB; from the line EA
produced take $EF = EB$. On AF make the ſquare AH.
Then is $AH = AB \times BG$.

For HG being drawn out to I; the rectangle $DH+$
$EAq$ (*c*) $=EFq$ (*d*) $=EBq$ (*e*) $=BAq+EAq$: Therefore c 6. 2.
is $DH$ (*f*) $=BAq=$ to the ſquare AC. Take away AI d *conſtr.*
common to both, there remains the ſquare $AH=GC$, e 47. 1.
that is, $AGq=AB \times BG$. *Which was to be done.* f 3. *ax.*

*Schol.*

This propoſition cannot be performed by numbers;
* for there is no number that can be ſo divided, that * 6. 13.
the product of the whole into one part ſhall be equal to
the ſquare of the other part.

### PROP. XII. Plate I. *Fig.* 56.

*In obtuſe-angled triangles as* ABC, *the ſquare that is made of the ſide* AC, *ſubtending the obtuſe angle* ABC, *is greater than the ſquares of the ſides* BC, AB, *that contain the obtuſe angle* ABC, *by a double rectangle contained under one of the ſides* BC, *which are about the obtuſe angle* ABC, *on which ſide produced the perpendicular* AD *falls, and under the line* BD, *taken without the triangle from the point on which the perpendicular* AD *falls to the obtuſe angle* ABC.

I ſay that $ACq = CBq+ABq+2CB \times BD$.

For theſe are all equal
$ACq$.
(*a*) $CDq+ADq$. a 47. 1.
(*b*) $CBq+2CBD+BDq+ADq$. b 4. 2.
$CBq+2CBD+$ (*c*) $ABq$. c 47. 1.

*Scholium.*

Hence, *the ſides of any obtuſe-angled triangle* ABC *being known, the ſegment* BD *intercepted betwixt the perpendicular* AD, *and the obtuſe angle* ABC, *as alſo the perpendicular it ſelf* AD, *ſhall be eaſily found out.*

Thus, Let AC be 10, AB 7, CB 5. Then is ACq 100, ABq 49, CBq 25. And $ABq+CBq=74$. Take that out of 100, then will 26 remain for 2 CBD. Wherefore CBD ſhall be 13; divide this by CB 5, there will $2\frac{3}{5}$ be found for BD. Whence AD will be found out by the 47. 1.

### PROP. XIII. *Fig.* 57.

*In acute-angled triangles as* ABC, *the ſquare made of the ſide* AB, *ſubtending the acute angle* ACB, *is leſs than the ſquares made*

*made of the ſides* AC, CB, *comprehending the acute angle* ACB, *by a double rectangle contained under one of the ſides* BC, *which are about the acute angle* ACB, *on which the perpendicular* AD *falls, and under the line* DC, *taken within the triangle from the perpendicular* AD, *to the acute angle* ACB.

I ſay that $ACq+BCq=ABq+2BCD$.

For theſe are equal.
$ACq+BCq$.
a 47. 1. (*a*) $ADq+DCq+BCq$.
b 7. 2. (*b*) $ADq+BDq+2BCD$.
c 47. 1. (*c*) $ABq+2BCD$.

*Coroll.*

Hence, *the ſides of an acute-angled triangle* ABC *being known, you may find out the ſegment* DC, *intercepted betwixt the perpendicular* AD, *and the acute angle* ACB, *as alſo the perpendicular it ſelf* AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from $ACq+BCq$, that is, from $225+196=421$. Then remains 252 for 2BCD, wherefore BCD will be 126, divide this by BC 14, then will 9 be found out for DC. From whence it follows, $AD=\sqrt{225-81}=12$.

## PROP. XIV. Plate I. *Fig.* 58.

*To find a ſquare* MC *equal to a right-lined figure given* A.

a 45. 1. (*a*) Make the rectangle $DB=A$, and produce the great-
b 10. 1. er ſide thereof DC to F, ſo that $CF=CB$, (*b*) biſect DF,
in G, about which as the center, at the diſtance of GF,
deſcribe the circle FHD, and draw out BC, till it meets
* 46. 1. the circumference in H. Then ſhall be $CHq={}^{*}MC$
c *conſtr.* $=A$.
d 5. 2. *and* For let GH be drawn. Then is A (*c*) $=$DB (*c*) $=$
3. *ax.* DCF (*d*) $=GFq-GCq$ (*e*) $=HCq$ (*c*) $=MC$. *Which*
e 47. 1. *and* *was to be done.*
3. *ax.*

*The End of the ſecond* BOOK.

The

Plate I. *Facing Pag.* 34.

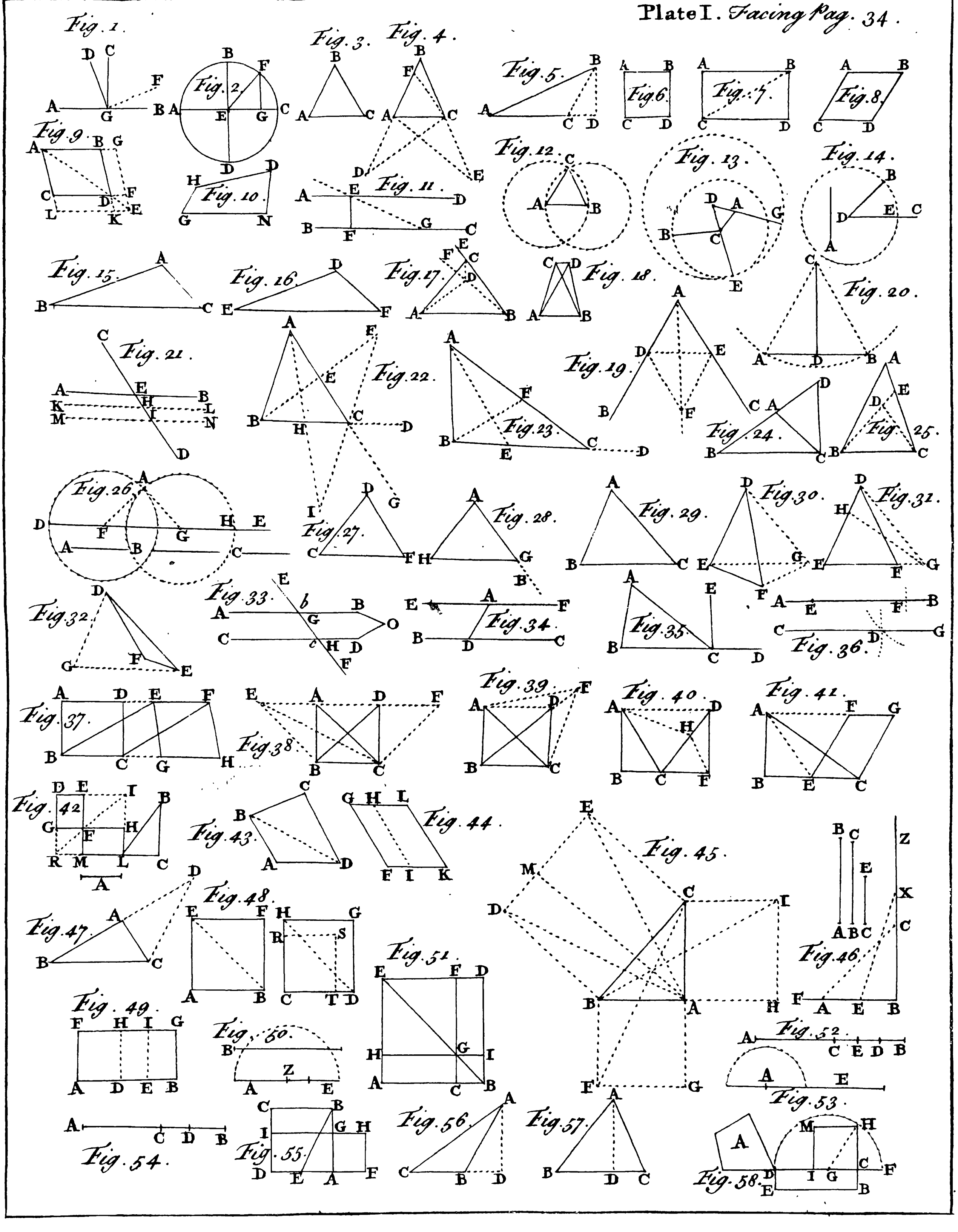


Fig. 1. D C F A G B
Fig. 2. B F A E G C D
Fig. 3. B A C
Fig. 4. B F A C D E
Fig. 5. B A C D
Fig. 6. A B C D
Fig. 7. A B C D
Fig. 8. A B C D
Fig. 9. A B G C D F E L K
Fig. 10. H D G N
Fig. 11. A E D B F G C
Fig. 12. C A B
Fig. 13. D A G B C E
Fig. 14. B D E C A
Fig. 15. A B C
Fig. 16. D E F
Fig. 17. E F C D A B
Fig. 18. C D A B
Fig. 19. A D E B F
Fig. 20. C A D B
Fig. 21. C A E B K H L M I N D
Fig. 22. A E E B C D H I G
Fig. 23. A F B E C D
Fig. 24. D C A B C
Fig. 25. A E D B C
Fig. 26. A D H E F G A B C
Fig. 27. D C F
Fig. 28. A H G B
Fig. 29. A B C
Fig. 30. D E G F
Fig. 31. D H E F G
Fig. 32. D F G E
Fig. 33. E b A G B O C c H D F
Fig. 34. E A F B D C
Fig. 35. A E B C D
Fig. 36. A E F B C D G
Fig. 37. A D E F B C G H
Fig. 38. E A D F B C
Fig. 39. F A D B C
Fig. 40. A D H B C F
Fig. 41. A F G B E C
Fig. 42. D E I B G F H R M L C A
Fig. 43. C B A D
Fig. 44. G H L F I K
Fig. 45. E M C I D B A H F G
Fig. 46. B C E A B C Z X C F A E B
Fig. 47. D A B C
Fig. 48. E F H G R S A B C T D
Fig. 49. F H I G A D E B
Fig. 50. B Z A E
Fig. 51. E F D G H I A C B
Fig. 52. A C E D B
Fig. 53. A E
Fig. 54. A C D B
Fig. 55. C B I G H D E A F
Fig. 56. A C B D
Fig. 57. A B D C
Fig. 58. A M H C F D I G E B

# The THIRD BOOK OF EUCLIDE'S ELEMENTS.

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## *Definitions.*

I. EQual circles (GABC, HDEF) are ſuch whoſe diameters are equal; or, from whoſe centers right-lines drawn GA, HD, are equal. Plate II. *Fig.* 1, 2.

II. A right-line MO, is ſaid to touch a circle GABC, when touching the ſame, and being produced, it cutteth it not. *Fig.* 2.

The right-line GCO cuts the circle GABC.

III. Circles are ſaid to touch one the other, which touch, but cut not one the other. As *Fig.* 8.

Thoſe of *Fig.* 7. cut each other.

IV. In a circle GABD, right-lines FE, KL, are ſaid to be equally diſtant from the center, when perpendiculars GH, GN, drawn from the center G to them, are equal. And that line BC is ſaid to be furtheſt diſtant from it, on which the greater perpendicular GI falls. *Fig.* 3.

V. A ſegment of a circle (BPC) is a figure contained under a right-line BC, and a portion of the circumference of a circle BPC. *Fig.* 2.

VI. An angle of a ſegment CBP, is that angle which is contained under a right-line BC, and the circumference of a Circle PB.

VII. An angle BPC is ſaid to be in a ſegment BPC, when in the circumference thereof ſome point P is taken, and from it right-lines BP, CP, drawn to the ends of the right-line BC, which is the baſe of the ſegment; then the angle BPC contained under the adjoined lines BP, CP, is ſaid to be an angle in a ſegment.

VIII. But when the right-lines BP, CP, comprehending the angle BPC, receive any periphery of the circle BAC, then the angle BPC is ſaid to ſtand upon that periphery.

*Fig.* 1. IX. A ſector of a circle (HRS) is when an angle RHS is ſet at the center H of that circle; namely, that figure RHS comprehended under the right-lines RH, SH, containing the angle, and the part of the circumference RS, intercepted between them.

*Fig.* 1, 2. X. Like ſegments of a circle (ABC, DFE) are thoſe which include equal angles (A B C, D E F;) or, in which the angles ABC, DEF, are equal.

### PROP. I. Plate II. *Fig.* 4.

*To find the center* F *of a given circle* ABC.

Draw a right-line AC any-wiſe in the circle, which biſect in E; thro' E draw a perpendicular DB, and biſect the ſame in F; the point F ſhall be the center.

If you deny it, let G, a point without the line BD, be
the center (for it cannot be in the line BD, ſince that is
divided unequally in every point but F) let the lines
GA, GC, GE, be drawn. Now if G be the center, (*a*)
a 15. *def.* 1. then is GA=GC, and AE=EC, by conſtruction, and
b 8. 1. the ſide GE common. (*b*) Therefore are the angles GEA,
c 10. *def.* 1. GEC, equal, and (*c*) conſequently right. (*d*) Therefore
d 12. *ax.* the angle GEC=FEG. (*e*) *Which is abſurd.*
e 9. *ax.*

*Coroll.*

Hence, if in a circle a right-line BD biſect any right-line AC at right-angles, the center ſhall be in the cutting line BD.

*Fig.* 5. *Andr. Tacq.* *The center of a circle is eaſily found out by applying the top of a ſquare to the circumference thereof.* For if the right-line GH that joins the points G, H, in which the ſides of the ſquare G B, B H, cut the circumference, be biſected in E, the point E ſhall be the center. The demonſtration whereof depends upon *Prop.* XXXI. of this Book.

### PROP. II. *Fig.* 2.

*If in the circumference of a circle* GAC, *any two points* A, C, *be taken, the right-line* AC, *which joins thoſe two points, ſhall fall within the circle.*

Take in the right-line AC any point S; from the cen-
a 15. *def.* 1. ter G draw GA, GS, GC. Becauſe GA (*a*) =GC,
b 5. 2. therefore is the angle A (*b*)=C. But the angle GSC (*c*)
c 16. 2. ⊏ A, therefore is GSC (*d*) ⊏ C, therefore GC (*d*) ⊏ GS.
d 19. 1. But

But GC only reaches the cirumference, therefore G S comes not ſo far; wherefore the point S is within the circle. The ſame may be proved of any other point in the line AC. And therefore the whole line AC falls within the circle. *Which was to be dem.*

### *Coroll.*

Hence, if a right-line touch a circle, without cutting it, it touches but in one point.

## PROP. III. Plate II. *Fig.* 5.

*If in a circle* EABC, *a right-line* BD *drawn thro' the center, biſects any other line* AC, *not drawn thro' the center, it ſhall alſo cut it at right-angles: And if cuts it at right-angles, it ſhall alſo biſect the ſame.*

From the center E let the lines EA, EC, be drawn.

1. *Hyp.* Becauſe AF (*a*) =FC, and EA (*b*) =EC, and a *hyp.*
the ſide EF common; the angles EFA, EFC, (*c*) ſhall b 15.*def.*1.
be equal, and (*d*) conſequently right. *Which was to be* c 8. 1.
*demonſtrated.* d 10. *def.*1.
2. *Hyp.* Becauſe EFA (*e*) =EFC, and the angle EAF e *hyp. and*
(*f*)=ECF, and the ſide EF common; (*g*) therefore is AF 12. *ax.*
=FC. Therefore AC is cut into two equal parts. f 5. 1.
*Which was to be demonſtrated.* g 26. 1.

### *Coroll.*

Hence, in any equilateral or Iſoſceles triangle, if a line drawn from the vertical angle biſect the baſe, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle biſects the baſe.

## PROP. IV. *Fig.* 6.

*If in a circle* ACD, *two right-lines* AB, CD, *cut each other, and neither of them paſs thro' the center* E, *they ſhall not cut each other into equal parts.*

For if one line paſs thro' the center, 'tis plain it cannot be biſected by the other; becauſe by hypotheſis, the other does not paſs thro' the center.

If neither of them paſs thro' the center, then from the center E draw EF; now if AB, CD, were both
biſected in F, then (*a*) would the angles EFB, EFD, be a 3. 3.
both right, and conſequently equal. (*b*) *Which is abſurd.* b 9. *ax.*

### PROP. V. Plate II. *Fig.* 7.

*If two circles* BAC, BDC, *cut each other, they ſhall not have the ſame center* E.

For otherwiſe the lines EB, EDA, drawn from E, the
a 15. *def.* 1. common center, would be DE (*a*) = EB (*a*) = EA (b)
b 9. *ax.* *Which is abſurd.*

### PROP. VI. *Fig.* 8.

*If two circles* BAC, BDE, *inwardly touch each other (in* B*) they have not one and the ſame center* F.

For otherwiſe the right-lines FB, FDA, drawn from
a 15. *def.* 1. the center F, would be FD (*a*) = FB (*a*) = FA. (*b*) *Which*
b 9. *ax.* *is abſurd.*

### PROP. VII. *Fig.* 9.

If in AB, the diameter of a circle, ſome point G be taken, which is not the center of the circle, and from that point certain right-lines GC, GD, GE, fall on the circle, the greateſt line ſhall be that (GA) which paſſeth thro' the center F; the leaſt, the remainder of the ſame line (GB.) And of all the other lines, the line GC, neareſt to that which was drawn thro' the center is always greater than any line farther removed GD; and there can but two equal lines fall from the ſame point on the circle, *viz.* one on each ſide of the leaſt GB, or of the greateſt GA.

From the center F draw the right-lines FC, FD, FE;
* 23. 1. * make the angle BFH = BFE.
a 20. 1. 1. GF + FC (that is GA) (*a*) $\sqsubset$ GC. *Which was to be demonſtrated.*

b 15. *def.* 2. The ſide FG is common, and FC (*b*) = FD, and
c 9. *ax.* the angle GFC (*c*) $\sqsubset$ GFD; (*d*) wherefore the baſe GC
d 24. 1. $\sqsubset$ GD.

e 20. 1. 3. FB (FE) (*e*) $\sqsupset$ GE + GF. Therefore FG, which
f 5. *ax.* is common, being taken away from both, there remains BG $\sqsupset$ EG.

4. The ſide FG is common, and FE = FH, and the
g *conſt.* angle BFH (*g*) = BFE; (*h*) Therefore is GE = GH.
h 4. 1. But that no other line GD from the point G, can be equal to GE, or GH, is already proved. *Which was to be demonſtrated.*

## PROP. VIII. Plate II. *Fig.* 10.

If ſome point A be taken without a circle, and from that point be drawn certain right-lines AI, AH, AG, AF, to the circle, and of thoſe one AI, be drawn thro' the center K, and the others any wiſe; of all thoſe lines that fall on the concave of the circumference, that is the greateſt (AI) which is drawn thro' the center; and of the others, that (AH) which is neareſt to the line that paſſes thro' the center, is greater than that which is more diſtant, AG. But of all thoſe lines that fall on the convex part of the circle, the leaſt is that (AB) which is drawn from the point A, to the diameter IB; and of the others, that (AC) which is neareſt to the leaſt, is leſs than that which is farther diſtant AD. And from that point there can be only two equal right-lines AC, AL, drawn, which ſhall fall on the circumference on each ſide of the leaſt line AB, or of the greateſt AI.

From the center K, draw the right-lines KH, KG, KF, KC, KD, KE, and make the angle AKL= AKC.

1. AI (AK+KH) (*a*) $\sqsubset$ AH. a 20. 1.

2. The ſide AK is common, and KH=KG, and the angle AKH $\sqsubset$ AKG; (*b*) therefore the baſe AH $\sqsubset$ AG. b 24. 1.

3. KA (*c*) $\sqsupset$ KC+CA. From hence take away KC, KB, which are equal; then will remain AB, (*d*) $\sqsupset$ AC. c 20. 1. d 5. *ax.*

4. AC+CK (*e*) $\sqsupset$ AD+DK. Take from both, CK, DK, which are equal; then remains AC (*f*) $\sqsupset$ AD. e 21. 1. f 5. *ax.*

5. The ſide KA is common, and KL=KC, and the angle AKL (*g*)=AKC; (*h*) therefore LA=CA. But that no other line could be drawn equal to theſe, was proved above. Therefore, &c. g *conſtr.* h 4. 1.

## PROP. IX. *Fig.* 11.

*If in a circle* BCK, *a point* A *be taken, and from that point more than two equal right-lines* AB, AC, AK, *can be drawn to the circumference, then is that point* A *the center of the circle.*

For (*a*) from no point without the center can more than two equal right-lines be drawn to the circumference. Therefore A is the center. *Which was to be demonſtrated.* a 7. 3.

## PROP. X. Plate II. *Fig.* 12.

A *circle* IAKBL, *cannot cut another circle* IEKFL, *in more than two points.*

Let one circle, if it may be, cut the other in three
points I, K, L, and IK, KL, being join'd, let them be
a *cor.* 1. 3. biſected in M and N. (*a*) Both circles have their centers
in their perpendiculars MC, NH, and in the interſection
of thoſe perpendiculars which is O. Therefore the circles that cut each other have the ſame center. Which is falſe, by *Prop.* 5. 3.

## PROP. XI. *Fig.* 8.

*If two circles* FBDE, GABC, *touch one the other inwardly, and their centers be taken* G, F; *a right-line* FG *joining their centers, and produced, ſhall cut the circumference in* B, *the point of contact of the circles,*

If it can be, let the right-line FG produced cut the
circles in ſome other point than B; ſo that not GFB,
but GFIK, ſhall be a right-line. Let the line FB be
a 15. *def.* 1. drawn. Now, becauſe FB (*a*) = FI, and FK (*b*) ⊐ GB
b 7. 3. (ſince the right-lines GFB paſſes thro' G, the center of
c 9. *ax.* the greater circle) therefore is FI ⊐ FK. (*c*) *Which is abſurd.*

## PROP. XII. *Fig.* 13.

*If two circles* ACD, BCE, *touch one the other outwardly, the right-line* AB, *which joins their centers* A, B, *ſhall paſs thro' the point of contact* C.

If it may be, let ADEB be a right-line cutting the
circles, not in the point of contact C, but in the points
a 20. 1. D, E; draw AC, CB, then is AD+EB (AC+CB) (*a*)
b 9. *ax.* ⊏ ADEB. (*b*) *Which is abſurd.*

## PROP. XIII. *Fig.* 8.

*A circle* GCB *cannot touch a circle* FBE *in more points than one* B, *whether it be inwardly or outwardly.*

1. Let one circle (if it can be) touch another in two
a 11. 3. points B, E, (*a*) Then will the right-lines GF, that joins
b 15. *def.* 1. the centers, if it be produced, fall as well in B as E.
c 15. *def.* 1. Now becauſe GE (*b*) = GB, and FE ⊏ GE, therefore
d 9. *ax.* is FB ((*c*) FE) ⊏ GB. (*d*) *Which is abſurd.*

2. If it be ſaid to touch outwardly in the points L
e 2. 3. and M, then draw the line LM, (*e*) which will be in both
circles.

circles. Therefore thoſe circles cut one the other; which is againſt the Hyp.

### PROP. XIV. Plate II. *Fig.* 14.

*In a circle* EABC, *equal right-lines* AC, BD, *are equally diſtant from the center* E : *And right-lines* AC, BD, *which are equally diſtant from the center, are equal among themſelves.*

From the center E, draw the perpendiculars EF, EG,
(*a*) which will biſect the lines AC, BD; join EA, EB. a 3. 3.
1. *Hyp.* AC = BD, therefore AF (*b*) = BG. But b 7. *ax.*
alſo EA = EB; therefore FEq (*c*) = EAq — AFq = c 47. 1.
EBq — BGq (*c*) = EGq. (*d*) Therefore FE = EG. *and* 3. *ax.*
2. *Hyp.* EF = EG. Therefore AFq (*e*) = EAq —
EFq = EBq — EGq = BGq. Therefore AF (*d*) = d *ſchol.*
GB, and (*e*) conſequently AC = BD. *Which was to be* 48. 1.
*demonſtrated.* e 6. *ax.*

### PROP. XV. *Fig.* 15.

*In a circle* GABC, *the greateſt line is the diameter* AD;
*and of all other lines, that* FE, *which is neareſt to the*
*center* G, *is greater than any line* BC *farther diſtant from it.* a 15. *def.* 1.
1. Draw GB, and GC. The diameter AD (*a*) GB b 20. 1.
+ GC (*b*) ⊏ BC.
2. Let the diſtance GI be ⊏ GH. Take GN =
GH. Thro' the point N draw KL, perpendicular to
GI: Join GK, GL. Becauſe GK = GB, and GL =
GC, and the angle KGL ⊏ BGC; (*c*) therefore is KL c 24. 1.
(FE) ⊏ BC. *Which was to be demonſtrated.*

### PROP. XVI. *Fig.* 16.

*A line* CD, *drawn from the extreme point of the diameter* HA, *of a circle* BALH, *perpendicular to the ſaid diameter, ſhall fall without the circle; and between the ſame right-line and the circumference, cannot be drawn another line* AL. *And the angle of the ſemicircle* BAI, *is greater than any right-lined acute angle* BAL; *and the remaining angle without the circumference* DAI, *is leſs than any right-lined angle.*

1. From the center B, to any point F, in the right-
line AC, draw the right-line BF. The ſide BF ſubtend-
ing the right-angle BAF, is (*a*) greater than the ſide BA, a 19. 1.
which is oppoſite to the acute-angle BFA. Therefore,
whereas BA, (BG) reaches to the circumference, BF,
ſhall reach further; and ſo the point F, and for the

ſame reaſon any other point of the line AC, ſhall be without the circle.

2. Draw BE perpendicular to AL. The ſide BA, op-
b 19. 1. poſite to the right-angle BEA, is (*b*) greater than the ſide BE, which ſubtends the acute-angle BAE; therefore the point E, and ſo the whole line EA, falls within the circle.

3. Hence it follows, that any acute-angle, to wit, EAD, is greater than the angle of contact DAI, and that any acute-angle BAL is leſs than the angle of a ſemicircle BAI. *Which was to be dem.*

*Coroll.*

Hence, a right-line drawn from the extremity of the diameter of a circle, and at right-angles, is a tangent to the ſaid circle.

From this propoſition are gathered many paradoxes, and wonderful conſectaries, which you may meet with in the interpreters.

## PROP. XVII. Plate II. *Fig.* 17.

*From a point given* A, *to draw a right-line* AC, *which ſhall touch a circle given* DBC.

From D, the center of the circle given, to the given point A, let the line DA be drawn, cutting the circumference in B, from the center D, deſcribe another circle thro' the point A; and from B, draw a perpendicular to AD, which ſhall meet with the circle AE in the point E; and draw ED meeting with the circle BC, in the point C. Then a line drawn from A to C, ſhall touch the circle DBC.

a 15. *def.* 1. For DB (*a*) = DC, and DE (*a*) = DA, and the angle
b 4. 1. D is common; (*b*) therefore the angle ACD = EBD
c *cor.* 16 3. and both right. (*c*) Therefore AC, touches the circle in C. *Which was to be done.*

## PROP. XVIII. *Fig.* 16.

*If any right-line* CA *touches a circle* BALH, *and from the center to the point of contact* A, *a right-line* BA *be drawn; that line* BA *ſhall be perpendicular to the tangent* CA.

If you deny it, let ſome other line BF be drawn from
a *def.* 2. 3. the center B, perpendicular to the tangent, and (*a*)
b *cor.* 17. 1. cutting the circle in G. Therefore, whereas the angle
c 19. 1. BFA is ſaid to be right, (*b*) thence the angle BAF
d 9. *ax.* acute; (*c*) ſo that BA (BG) $\sqsubset$ BF. (*d*) *Which is abſurd.*

PROP.

## PROP. XIX. Plate II. *Fig.* 16.

*If any right-line* CD *touch a circle, and from the point of contact* A, *a right-line* AH, *be erected at right-angles to the tangent, the center of the circle ſhall be in the line* AH *ſo erected.*

If you deny it, let the center be without the line AK,
in the point K; and from K, to the point of contact,
let KA be drawn. Therefore the angle KAD is right,
and (*a*) conſequently equal to the angle HAD, which a 12. *ax.*
was right by Hypotheſis. (*b*) *Which is abſurd.* b 9. *ax.*

## PROP. XX. *Fig.* 18, 19, 20.

*In a circle* DABC, *the angle* BDC *at the center, is double of the angle* BAC *at the circumference, when the ſame arch of the circle* BC, *is the baſe of the angle.*

Draw the Diameter ADE, The outward angle BDE a 32. 1.
(*a*)=DAB+DBA (*b*) =2DAB : In like manner the an- b 5. 1.
gle EDC=2DAC, Therefore in the firſt caſe (*c*) the c 2 *ax.*
whole angle BDC=2BAC, and in the third caſe the d 20. *ax.*
remaining angle BDC (*d*) =2BAC. *Which was to be demonſtrated.*

## PROP. XXI. *Fig.* 21.

*In a circle* EDAC, *the angles* DAC, *and* DBC, *which are in the ſame ſegment, are equal one to the other.*

1. *Caſe.* If the ſegment DABC be greater than a
ſemicircle, from the center E draw ED, EC. Then is
twice the angle A (*a*) = E (*a*) = 2 B. *Which was to be de-* a 20. 3.
*monſtrated.*

2. *Caſe.* If the ſegment, (*Fig.* 22.) be leſs than a ſemicir-
cle, then is the ſum of the angles of the triangle ADF
equal to the ſum of the angles of the triangle BCF; from
each let AFD equal to BFC, (*b*) and ADB (*c*) = ACB, b 15. 1.
be taken away, then remains DAC=DBC. *Which was* c *by the* 1ſt
*to be demonſtrated.* *caſe.*

## PROP. XXII. *Fig.* 23.

*The angles* ADC, ABC, *of a quadrilateral figure* ABCD, *deſcribed in a circle, which are oppoſite one to the other, are equal to two right-angles.*

Draw AC, BD. The angle ABC+BCA+BAC (*a*) a 32. 1.
= 2 right-angles. But BDA (*b*) =BCA, and BDC (*b*) b 21. 3.
=BAC. (*c*) Therefore ABC+ADC=2 right-angles. c 1. *ax.*
*Which was to be demonſtrated.*

Coroll

*Corol.*

1. Hence, if one ſide AB of a quadrilateral, deſcribed in a circle, be produced, the external angle EBC is equal to the internal angle ADC, which is oppoſite to that ABC, and adjacent to EBC, as appears by 13. 1. and 3. *ax.*

2. A circle cannot be deſcribed about a Rhombus becauſe its oppoſite angles are greater, or leſs than two right-angles.

*Schol.* Plate II. *Fig.* 24.

*If in a quadrilateral* ABCD, *the angles* A, *and* C, *which are oppoſite, be equal to two right-angles, then a circle may be deſcribed about that quadrilateral.*

For a circle will paſs through any three angles B,C,D, (as appears by 5. 4.) I ſay that it ſhall alſo paſs thro'
A, the 4th angle of ſuch a quadrilateral: For if you
a 22. 3. deny it, let the circle paſs thro' F: Therefore the right-
b *hyp.* lines BF, FD, BD being drawn, the angle C+F (*a*)=
c 3. *ax.* 2 right (*b*) = C+A; wherefore A (*c*) is equal to F, (*d*)
d 21. 1. *Which is abſurd.*

## PROP. XXIII. *Fig.* 25.

*Two like and unequal ſegments of circles* ABC, ADC, *cannot be ſet on the ſame right-line* AC, *and on the ſame ſide thereof.*

For if they are ſaid to be like, draw the line CB
cutting the circumference in D and B, join AB and AD.
a 10. *def.* 3. Becauſe the ſegments are ſuppoſed like, (*a*) therefore is
b 16. 1. the angle ADC=ABC. (*b*) *Which is abſurd.*

## PROP. XXIV. *Fig.* 26.

*Like ſegments, of circles* ABC, DEF, *upon equal right-lines* AC, DF, *are equal one to the other.* The baſe AC being laid on the baſe DF, will agree with it, becauſe AC=
DF. Therefore the ſegment ABC ſhall agree with the
ſegment DEF (for otherwiſe it ſhall fall either within or
a 23. 3. without) and if ſo (*a*) then the ſegments are not like,
which is contrary to the Hypotheſis, and at leaſt it ſhall
fall partly within and partly without, and ſo cut in three
b 10. 3. points, (*b*) which is abſurd. (*c*) Therefore the ſegment
c 8. *ax.* ABC=DEF. *Which was to be demonſtrated.*

PROP.

## PROP. XXV. Plate II. *Fig.* 27.

*A ſegment of a circle* ABC, *being given, to deſcribe the whole circle whereof that is a ſegment.*

Let two right-lines be drawn AB, BC, which biſect in the points D and E. From D and E draw the perpendiculars DF, EF, meeting in the point F. I ſay this point ſhall be the center of the circle.

For the center ſhall be as well in (*a*) DF as EF, a *cor.* 1. 3.
therefore it muſt be in the point F, which is common to them both. *Which was to be done.*

## PROP. XXVI. *Fig.* 28, 29.

*In equal circles* GABC, HDEF, *equal angles ſtand upon equal parts of the circumference,* AC, DF; *whether thoſe angles be made at the centers* G, H, *or at the circumferences,* B, E.

Becauſe the circles are equal, therefore is $GA=HD$,
and $GC=HF$; alſo by Hypotheſis the angle $G=H$, a 4. 1.
(*a*) therefore $AC=DF$. Moreover the angle B (*b*) $=\frac{1}{2}G$ b 20. 3.
(*c*) $=\frac{1}{2}H$ (*b*) $=E$. (*d*) Therefore the ſegments ABC, c *hyp.*
DEF are like, and (*e*) conſequently equal; (*f*) whence d 10. *def.* 3.
the remaining ſegments alſo AC, DF, are equal. *Which* e 24. 3.
*was to be demonſtrated.* f 3. *ax.*

### *Schol. Fig.* 30.

In a circle ABCD, let an arch AB be equal to DC;
then ſhall AD be parallel to BC. For the right-line AC
being drawn, the angle ACB (*a*) $=$ CAD; wherefore a 26. 3.
by 27. 1. the ſaid ſides are parallel.

## PROP. XXVII. *Fig.* 28, 29.

*In equal circles* GABC, HDEF, *the angles ſtanding upon equal parts of the circumference.* AC, DF, *are equal between themſelves, whether they be made at the centers* G, H, *or at the circumferences,* B, E.

For if it be poſſible, let one of the angles AGC be a 26. 3.
$\sqsubset$ DHF, and make $AGI=DHF$; thence is the arch b *hyp.*
AI (*a*) $=$ DF (*b*) $=$ AC. (*c*) *Which is abſurd.* c 9. *ax.*

### *Schol. Fig.* 31.

*A right-line* EF, *which being drawn from* A *the middle point of any periphery* BC, *touching the circle, is parallel to* *to the right-line* BC, *ſubtending the ſaid periphery,*

From the center D draw a right-line DA to the point of contact A, and join DB, DC.

The ſide GD is common, and DB=DC, and the
a 27. 3. angle BDA (*a*) =CDA, (becauſe the arches BA, CA are
b *hyp.* (*b*) equal) therefore the angles at the baſe DGB, DGC
c 4. 1. are (*c*) equal, and (*d*) conſequently right; but the inward
d 10. *def.* 1. angles GAE, GAF are alſo (*e*) right; (*f*) therefore BC,
e *hyp.* EF are parallel. *Which was to be demonſtrated.*
f 28. 1.

### PROP. XXVIII. Plate II. *Fig.* 28, 29.

*In equal circles* GABC, HDEF, *equal right-lines* AC, DF, *cut off equal parts of the circumference, the greateſt* ABC, *equal to the greateſt* DEF, *and the leaſt* ALC *to the leaſt* DKF.

From the centers G, H, draw GA, GC, and HD, HF.
a *hyp.* Becauſe GA=HD, and GC=HF, and AC (*a*)=DF,
b 8. 1. (*b*) therefore is the angle G=H; (*c*) whence the arch
c 26. 3. ALC=DKF; (*d*) and ſo the remaining arch ABC=
d 3. *ax.* DEF. *Which was to be demonſtrated.*

But if the ſubtended line AC be ⊏ or ⊐ than DF, then in like manner will be the arch AC be ⊏ or ⊐ than DF.

### PROP. XXIX. *Fig.* 28, 29.

*In equal circles* GABC, HDEF, *the right-lines* AC, DF, *which ſubtend equal peripheries* ABC, DEF, *are equal.*

Draw the lines GA, GC, and HD, HF. Becauſe
GA = HD, and GC = HF, and (becauſe the arches
a *hyp.* AC, DF are (*a*) equal) the angle G (*b*) = H, (*c*) there-
b. 27. 3. fore is the baſe AC = DF. *Which was to be demonſtrated.*
c 4. 1.

This and the three precedent propoſitions may be underſtood alſo of the ſame circle.

### PROP. XXX. *Fig.* 32.

*To cut a Periphery given* ABC *into two equal parts.*

Draw the right-line AC, and biſect it in D; from D draw a perpendicular DB meeting with the arch in B, it ſhall biſect the ſame.

For join AB, and CB. The ſide DB is common,
a *conſtr.* and AD (*a*) = DC, and the angle ADB (*b*) = CDB.
b 12. *ax.* (*c*) Therefore AB = BC; (*d*) whence the arch AB=BC.
c 4. 1. *Which was to be done.*
d 28. 3.

## PROP. XXXI. Plate II. *Fig.* 33.

*In a circle the angle* ABC, *which is in the ſemicircle, is a right-angle; but the angle, which is in the greater ſegment* BAC, *is leſs than a right-angle, and the angle which is in the leſſer ſegment* BFC *is greater than a right-angle. Moreover, the angle of the greater ſegment is greater than a right-angle, and the angle of the leſſer ſegment is leſs than a right-angle.*

From the center D draw DB. Becauſe DB = DA,
therefore is the angle A (*a*) = DBA, and the angle DCB a 5. 1.
(*a*) = DBC, (*b*) therefore the angle ABC = A + b 2. *ax.*
ACB, (*c*) = EBC, (*d*) ſo that ABC and EBC are right- c 32. 1.
angles. *W. W. D.* (*e*) Therefore BAC is an acute-an- d 10. *def.* 1.
gle. *W. W. D.* And further, whereas BAC + BFC (*f*) e *cor.* 17. 1.
= 2 right-angles, therefore BFC is an obtuſe-angle. f 22. 3.
Laſtly, the angle contained under the right-line CB, and
the arch BAC is greater than the right-angle ABC;
but the angle made by the right-line CB, and the pe-
riphery of the leſſer ſegment BFC (*g*) is leſs than the g 9. *ax.*
right-angle EBC. *Which was to be demonſtrated.*

### *Schol.*

*In a right-angled triangle* ABC, *if the hypothenuſe* (*or line ſubtending the right-angle*) AC *be biſected in* D, *a circle drawn from the center* D, *through the point* A, *ſhall alſo paſs through the point* B; *as you may eaſily demonſtrate from this* prop. and 21. 1.

## PROP. XXXII. *Fig.* 34.

*If a right-line* AB *touch a circle, and from the point of contact be drawn a right-line* CE, *cutting the circle, the angles* ECB, ECA, *which it makes with the tangent line, are equal to thoſe angles* EDC, EFC, *which are made in the alternate ſegments of the circle.*

Let CD, the ſide of the angle EDC be perpendi-
cular to AB, (*a*) for it is to the ſame purpoſe, (*b*) there- a 26. 3.
fore CD is the diameter, (*c*) therefore the angle CED in a b 19. 3.
ſemicircle is a right-angle, (*d*) and therefore the angle c 31. 3.
D + DCE = to a right-angle (*e*) = ECB + DCE. (*f*) d 32. 1.
Therefore the angle D = ECB, *Which was to be dem.* e *conſtr.*

Now whereas the angle ECB + ECA (*g*) = 2 right- f 3. *ax.*
angles (*h*) = D + F, from both of theſe take away g 13. 1.
ECB and D, which are equal, (*k*) there remains ECA h 22. 3.
= F. *Which was to be dem.* k 3. *ax.*

## PROP. XXXIII. Plate II. *Fig.* 35.

*Upon a right-line* AB *to deſcribe a ſegment of a circle* AIEB *which ſhall contain an angle* AIB, *equal to a right-lined angle given* C.

a 23. 1. (*a*) Make the angle BAD=C. Through the point A
draw the line AE perpendicular to HD. At the other
end of the line given AB make an angle ABF=BAF,
one of the ſides of which ſhall cut the line AE in F;
from the center F, through the point A, deſcribe a
circle, which ſhall paſs through B. (Becauſe the angle
c *conſtr.* FBA (*b*) =FAB, and (*c*) therefore FB=FA) AIB is
c 6. 1. the ſegment ſought. For becauſe HD is perpendicular
d *cor.* 16. 3. to the diameter AE, therefore HD (*d*) touches the circle
a 32. 3. which AB cuts. And therefore the angle AIB (*e*) =
f *conſtr.* BAD (*f*) = C. *Which was to be done.*

## PROP. XXXIV. *Fig.* 36.

*From a circle given* ABC *to cut off a ſegment* ABC *containing an angle* B, *equal to a right-lined angle given* D.

a 17. 3. (*a*) Draw a right-line EF which ſhall touch the circle
b 23. 1. given in A, (*b*) let AC be drawn making an angle FAC
c 32. 3. =D. Alſo draw (*b*) AB, making the angle BAE=D,
d *conſtr.* and join B, C. The line AB ſhall cut off the ſegment
ABC containing an angle B (*c*) =CAF (*d*) =D. *Which was to be done.*

## PROP. XXXV. *Fig.* 37.

*If in a circle* DBCA *two right-lines* AB, DC *cut each other, the rectangle comprehended under the ſegments* AE, EB, *of the one, ſhall be equal to the rectangle comprehended under the ſegments* CE, ED *of the other.*

1. *Caſe.* If the right-lines cut one the other in the center, the thing is evident.

2. *Caſe.* If one line AB (*Fig.* 38.) paſſes thro' the center
a 5. 2. F, and biſects the other line CD, then draw FD. Now
b *ſch.* 48. 1. the rectangle AEB+FEq (*a*) =FBq. (*b*) =FDq. (*c*) =
c 47. 1. EDq+FEq (*d*) =CED+FEq. (*e*) Therefore the rectan-
d *hyp.* gle AEB=CED. *Which was to be demonſtrated.*
e 3. *ax.* 3. *Caſe* If one of the lines AB (*Fig.* 39.) be the diameter, and cut the other line CD unequally, biſect CD by FG, a perpendicular from the center.

Theſe are equal.
The rectangle AEB+FEq.
*(f)* FBq (FDq) f 5. 2.
*(g)* FGq+GDq g 47. 1.
FGq+ *(h)* GEq+Rectang. CED. h 5. 2.
*(k)* FEq+CED. k 47. 1.

*(l)* Therefore the rectangle AEB=CED. l 3. *ax.*

4. *Caſe.* If neither of the right-lines AB, CD paſs thro'
the center, then through the point of interſection E, draw *Fig.* 40.
the diameter GH. By that which hath been already
demonſtrated it appears, that the rectangle AEB=GEH
=CED. *Which was to be demonſtrated.*

More eaſily, and generally, thus; join AC and BD, *Fig.* 41.
then becauſe the angles *(a)* CEA, DEB, and *(b)* alſo C, B a 15. 1.
(upon the ſame arch AD) are equal, thence are the tri- b 21. 3.
angles CEA, BED, *(c)* equiangular. *(d)* Wherefore CE : c *cor.* 32. 1.
EA : : EB : ED, and *(e)* conſequently CE × ED=AE × d 4. 6.
EB. *Which was to be demonſtrated.* e 16. 6.

The citations out of the 6 Book, both here and in the following *prop.* have no dependance on the ſame; ſo that it was free to uſe them.

## PROP. XXXVI. Plate II. *Fig.* 42.

*If any point be taken without a circle* EBC, *and from that point two right-lines* DA, DB, *fall upon the circle, whereof one* DA *cuts the circle, the other* DB *touches it, the rectangle comprehended under the whole line* DA *that cuts the circle, and* DC, *that part which is taken from the point given* D *to the convex of the periphery, ſhall be equal to the ſquare made of the tangent line.*

1. *Caſe.* If the ſecant AD paſſes thro' the center, then a 18. 3.
join EB, this *(a)* will make a right-angle with the line b 47. 1.
DB, wherefore DBq+ *(d)* EBq (ECq,) *(b)* =EDq *(c)* = c 6. 2.
AD × DC+ECq. Therefore AD × DC=DBq. *Which* d 3. *ax.*
*was to be demonſtrated.*

2. *Caſe.* But if AD paſſes not thro' the center, then *Fig.* 43.
draw EC, EB, ED, and EF perpendicular to AD, *(a)* a 3. 3.
wherefore AC is biſected in F.

Becauſe BDq+EBq *(b)* =DEq *(b)* EFq+FDq *(c)* = b 47. 1.
EFq+ADC+FCq *(d)* =ADC+CEq (EBq.) *(e)* There- c 6. 2.
fore is BDq=ADC. *Which was to be demonſtrated.* d 47. 1.

More eaſily, and generally thus; draw AB and BC. e 3. *ax.*
Then, becauſe the angles A, and DBC *(a)* are equal, *Fig.* 44.
and the angle D common to both; thence are the trian- a 32. 3.
gles BDC, ADB *(b)* equiangular. *(c)* Wherefore AD : b 32. 1.
DB : : DB : CD; and *(d)* conſequently AD×DC=DBq: c 4. 6.
*Which was to be demonſtrated.* d 17. 6.

*Coroll.* Plate II. *Fig.* 45.

1. Hence, If from any point A, taken without a circle, there be ſeveral lines AB, AC drawn which cut the circle; the rectangle comprehended under the whole lines AB, AC, and the outward parts AE, AF, are equal between themſelves.

For if the tangent AD be drawn, then is CAF=ADq
a 36. 3. (*a*) =BAE.

2. It appears alſo from hence, that if two lines AB, AC, (*Fig.* 46.) drawn from the ſame point do touch a circle, thoſe two lines are equal one to the other.

For if AE be drawn cutting the circle, then is ABq
a 36. 3. (*a*) = EAF (*b*) =ACq,

b 36. 3. 3. It is alſo evident, that from a point A, taken without a circle, there can be drawn but two lines AB, AC, that ſhall touch the circle.

For if a third line AD be ſaid to touch the circle,
c 2. *cor.* thence is AD (*c*) =AB (*c*) =AC. (*d*) *Which is abſurd.*

d 8. 3. 4. And, on the contrary, it is plain, that if two equal right-lines AB, AC, fall from any point A, upon the convex periphery of a circle, and that if one of theſe equal lines AB touch the circle, then the other AC touches the circle alſo.

e 2. *cor.* For if poſſible, let not AC, but another line AD,
f *hyp.* touch the circle; therefore is AD (*e*) =AC (*f*) =AB.
g 8. 3. (*g*) *Which is abſurd.*

## PROP. XXXVII. *Fig.* 47.

*If without a circle* EBF *any point* D *be taken, and from that point two right-lines* DA, DB *fall on the circle, whereof one line* DA *cuts the circle, the other* DB *falls upon it; and if alſo the rectangle comprehended under the whole line that cuts the circle, and under that part of it* DC *which is taken betwixt the point* D *and the convex periphery, be equal to that ſquare which is made of the line* DB *falling on the circle, I ſay that that line* DB *ſo falling ſhall touch the circle given.*

a 17. 3. From the point D (*a*) let a tangent DF be drawn, and
b *hyp.* from the center E draw ED, EB, EF. Now becauſe
c 36. 3. DBq (*b*)=ADC (*c*) =DFq, therefore is DB (*d*)=DF:
d 1. *ax.* & But EB=EF, and the ſide ED common; (*e*) therefore
*ſch.* 4. 1. the angle EBD=EFD; but EFD is a right-angle, and
e 8. 1. (*f*) therefore EBD is right alſo; and (*g*) therefore DB
f 12. *ax.* touches the circle. *Which was to be dem.*

g *cor.* 16. 3. *Coroll.*

h 8. 1. From hence it follows that the (*h*) angle EDB=EDF.

## *The End of the third* BOOK.

# The FOURTH BOOK OF EUCLIDE'S ELEMENTS.

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## *Definitions.*

I. A Right-lined figure is ſaid to be inſcribed in a right-lined figure, when every one of the angles of the inſcribed figure touch every one of the ſides of the figure wherein it is inſcribed.

*So the triangle* D E F *is inſcribed in the triangle* ABC. Plate II. *Fig.* 48.

II. In like manner a figure is ſaid to be deſcribed about a figure, when every one of the ſides of the figure circumſcribed, touch every one of the angles of the figure about which it is circumſcribed.

*So the triangle* ABC *is deſcribed about the triangle* DEF.

III. A right-lined figure is ſaid to be inſcribed in a circle, when all the angles of that figure which is inſcribed touch the circumference of the circle. As *Fig.* 50.

IV. A right-lined figure is ſaid to be deſcribed about a circle, when all the ſides of the figure which is circumſcribed touch the periphery of the circle. As *Fig.* 49.

V. After the like manner a circle is ſaid to be inſcribed in a right-lined figure, when the periphery of the circle touches all the ſides of the figure, in which it is inſcribed. *Fig.* 49.

VI. A circle is ſaid to be deſcribed about a figure when the periphery of the circle touches all the angles of the figure, which it circumſcribes.

VII. A right-line is ſaid to be fitted or applied in a circle when the extremes thereof fall upon the circumference; *as the right-line* AB. *Fig.* 6.

### PROP. I. *Fig.* 51.

*In a circle given* ABC *to apply a right-line* AB *equal to a right-*

*right-line given* D, *which doth not exceed* AC, *the diameter of the circle.*

a 3. *poſt.* & From the center A at the the diſtance AE=D (*a*) de-
3. 1. ſcribe a circle meeting with the circle given in B, draw
b 15. *def.* 1. AB. Then is AB (*b*) =AE (*c*)=D. *Which was to be*
c *conſtr.* *done.*

### PROP. II. Plate II. *Fig.* 49, 50.

*In a circle given* ABC *to deſcribe a triangle* ABC, *equiangular to a triangle given* DEF.

a 17. 3. Let the right-line GH (*a*) touch the circle given in A;
b 23. 1. (*b*) make the angle HAC=E, (*b*) and the angle GAB=F,
then join BC; and the thing is done.
c 32. 3. For the angle B (*c*)=HAC (*d*)=E, and the angle C
d *conſtr.* (*c*) =GAB (*d*) =F; (*e*) whence alſo the angle BAC=D.
e 32. 1. Therefore the triangle BAC inſcribed in the circle is
equiangular to DEF. *Which was to be done.*

### PROP. III. *Fig.* 52, 53.

*About a circle given* IABC *to deſcribe a triangle* LNM, *equiangular to a triangle given* DEF.

a 23. 1. Produce the ſide EF on both ſides; at the center I (*a*)
make an angle AIB=DEG, and an angle BIC=DFH.
Then through the points A, B, C, let three right-lines
b 17. 3. LN, LM, NM, (*b*) be drawn, touching the circle, and the
thing is done.
For it's evident that the right-lines LN, LM, MN,
c 13. *ax* will meet and make a triangle, (*c*) becauſe the angles
d 11. 3. LAI, LBI are right; ſo that if the (*d*) right-line AB
was drawn, it would make the angles LAB, LBA, leſs
than two right-angles.
e *ſch*, 32. 1. Since therefore the angle AIB+L (*e*) = 2 right-angles
f 13. *ax.* (*f*) =DEG+DEF, and AIB (*g*) =DEG; (*h*) therefore
g *conſtr.* is the angle L=DEF. By the ſame way of reaſoning
h 3. *ax.* the angle M=DFE. (*k*) Therefore alſo the angle N=
k 32. 1 D. And therefore the triangle LNM, deſcribed about
the circle, is equiangular to EDF, the triangle given.
*Which was to be done.*

### PROP. IV. *Fig.* 54.

*In a triangle given* ABC, *to deſcribe a circle* EFG.

a 9. 1 (*a*) Biſect the angles B and C with the right-lines BD,
b 12. 1. CD, meeting in the point D, (*b*) and draw the perpendiculars DE, DF, DG. A circle deſcribed from the
center

center D through E, will paſs through G and F, and touch the three ſides of the triangle.

For the angle DBE (*c*) =DBF; and the angle DEB (*d*) c *conſtr.*
=DFB; and the ſide DB common, (*e*) therefore DE= d 12 *ax,*
DF. For the ſame reaſon DG=DF, The circle there- e 26. 1.
fore deſcribed from the center D paſſes through the three points E, F, G; and whereas the angles at E, F, G, are right, therefore it touches all the ſides of the triangle. *Which was to be done.*

### *Schol.*

Hence, *The ſides of a triangles being known, their ſeg-* *Pet. Herig.*
*ments which are made by the touching of the circle inſcribed, ſhall be found,* Thus;

Let AB be 12, AC 18, BC 16, then is AB+BC=28. Out of which ſubduct 18=AC=AE+FC, there remains 10=BE+BF. Therefore BE, or BF = 5; and conſequently F C, or C G = 11. Wherefore GA, or AE, = 7.

## PROP. V. Plate II. *Fig.* 55, 56.

*About a triangle given* ABC, *to deſcribe a circle* FABC.

(*a*) Biſect any two ſides BA, CA, with perpendiculars a 10. &
DF, EF, meeting in the point F. I ſay this ſhall be 11. 1.
the center of the circle.

For, let the right-lines FA, FB, FC be drawn. Now
becauſe AD (*b*) =DB, and the ſide DF common, and b *conſtr.*
the angles FDA (*c*) =FDB, therefore is FB (*d*) = FA. c *conſtr.* &
After the ſame manner is FC=FA. Therefore a circle 12. *ax.*
deſcribed from the center F ſhall paſs through the an- d 4. 1.
gles of the triangle given (*viz.*) B, A, C. *Which was to be done.*

### *Coroll.*

* Hence, if a triangle be acute-angled, the center * 31. 3.
ſhall fall within the triangle; if right-angled, in the ſide oppoſite to the right-angle, and if obtuſe-angled, without the triangle.

### *Schol.*

By the ſame method may a circle be deſcribed, that ſhall paſs through three points given, not being in the ſame ſtrait-line.

## PROP. VI. Plate II. *Fig.* 57.

*In a circle given* EABCD *to inſcribe a ſquare* ABCD.

a 11. 1. (*a*) Draw the diameters AC, BD cutting each other at
right-angles in the center E. Join the extremes of theſe diameters with the right-lines AB, BC, CD, DA. *And the thing is done.*

b 26. 3. Now becauſe the four angles at E are right, the (*b*)
c 29. 3. arches and (*c*) ſubtended lines AB, BC, CD, DA, are
d 31. 3. equal; therefore is the figure ABCD equilateral, and
e 29. *def.* 1. all the angles in ſemicircles, and ſo (*d*) right. (*e*) Therefore ABCD is a ſquare inſcribed in a circle given. *Which was to be done.*

## PROP. VII. *Fig.* 58.

*About a circle given* EABCD, *to deſcribe a ſquare* FHIG.

Draw the Diameters AC, BD, cutting one the other at right-angles; through the extremes of theſe diameters
a 17. 3. (*a*) draw tangents meeting in F, H, I, G. *And the thing is done.*

b 18. 3. For becauſe (*b*) the angles A and C are right, (*c*) there-
c 28. 1. fore is FG parallel to HI. After the ſame manner is F H parallel to GI, and therefore F H I G is a Pgr. and alſo right-angled. It is equilateral becauſe FG (*d*)
d 34. 1. = HI (*d*) = DB (*e*) = CA (*d*) = FH (*d*) = GI.
e 15. *def.* 1. Wherefore FHIG is a (*f*) ſquare circumſcribed to the
f 10. *def.* 1. circle given. *Which was to be done.*

### *Schol.* *Fig.* 59.

A ſquare ABCD deſcribed about a circle is double of the ſquare EFGH inſcribed in the ſame circle.

For the rectangle HB = 2 HEF and HD = 2 HGF, by the 41. 1.

## PROP. VIII. *Fig.* 58.

*In a ſquare given* FGHI, *to inſcribe a circle* EABCD.

Biſect the ſides of the ſquare in the points B, D, A, C, cutting one the other in E, a circle drawn from the center E, thro' A, ſhall be inſcribed in the ſquare.

a 7. *ax.* For becauſe FA and HC are (*a*) equal and (*b*) parallel,
b *hyp.* (*c*) therefore is FH parallel to AC, parallel to GI. Af-
c 33. 1. ter the ſame manner is FG parallel to BD, parallel to HI; therefore EF, EG, EH, EI, are parallelograms.

Therefore FA (*d*) =FB (*e*) =AE=BE=CE=ED. The d 7. *ax.*
circle therefore deſcribed from the center E, through A e 34. 1.
ſhall paſs through A, B, C, D, and touch the ſides of the ſquare, ſince the angles A, B, C, D, are right. *Which was to be doee.*

## PROP. IX. Plate II. *Fig.* 57.

*About a ſquare given* ABCD, *to deſcribe a circle* EABCD.

Draw the diagonals AC, BD, cutting one the other in E. From the center E through A deſcribe a circle; I ſay this circle is circumſcribed to the ſquare.

For the angles ABD and BAC are (*a*) half of right- a 4. *cor.*
angles; (*b*) therefore EA=EB. After the ſame manner 32. 1.
is EA = ED=EC. The circle therefore deſcribed from b 6. 1.
the center E paſſes through A, B, C, D the angles of the ſquare given. *Which was to be done.*

## PROP. X. *Fig.* 60.

*To make an Iſocelcs triangle* ABD, *having each angle at the baſe* B, *and* ADB *double to the remaining angle* A.

Take any right-line AB, and divide it in C, (*a*) ſo that a 11. 2.
AB×BC may be equal to ACq. From the center A thro'
B, deſcribe the circle ABD; and in this circle (*b*) apply b 1. 4.
BD=AC, and join AD; I ſay ABD is the triangle
required. c 5. 4.

For draw DC, and through the points C, D, A, (*c*) d 37. 3.
draw a circle. Now becauſe AB×BC=ACq=BDq, (*d*) e 32. 3.
it is evident that BD touches the circle ACD which f 2. *ax.*
CD cutteth; (*e*) therefore is the angle BDC=A, and g 32. 1.
therefore the angle BDC+CDA (*f*) =A+CDA (*g*) = h 5. 1.
BCD. But BDC+CDA = BDA (*h*) = CBD; (*k*) k 1. *ax.*
therefore the angle BCD = CBD, and therefore DC (*l*) l 6. 1.
=DB= (*m*) AC; (*n*) wherefore the angle CDA = A m *conſtr.*
=BDC, therefore ADB = 2 A = ABD. *Which was* n 5. 1.
*to be done.*

### *Coroll.*

Whereas all the angles A, B, D, (*h*) make up two h 32. 1.
right-angles, it's evident that A is one fifth of two right-angles.

## PROP. XI. *Fig.* 61, 62.

*In a circle given* A B C D E *to inſcribe a Pentagon* ABCDE *equilateral and equiangular.*

a 10. 4. (*a*) Deſcribe an Iſoſceles triangle FGH, having each
b 2. 4. angle at the baſe double to the other; to the circle, (*b*) inſcribe a triangle C A D equiangular to the ſaid tri-
c 9. 1. angle F G H. (*c*) Biſect the angles at the baſe A C D and A D C with the right-lines DB, CE meeting with the circumference in B and E, join the right-lines CB, BA, AE, ED. *Then I ſay it is done.*

For it is evident by conſtruction, that the angles CAD, CDB, BDA, DCE, ECA, are equal; where-
d 26. 3. fore the (*d*) arches and (*e*) the lines ſubtending them DC,
e 29. 3. CB, BA, AE, DE, are equal. Therefore the penta-
f 27. 3. gon is equilateral, and equiangular, (*f*) becauſe the an-
g 2. *ax.* gles of it BAE, AED, &c. ſtand on equal (*g*) arches BCDE, ABCD, &c.

A more eaſy practice of this problem ſhall be deliver'd at 10. 13.

*Coroll.*

Hence, each angle of an equilateral and equiangular pentagon is equal to three-fifths of two right-angles, or ſix-fifths of one right-angle.

*Schol. Fig.* 63.

*Pet. Herig.* Generally all figures whoſe number of ſides is odd, are inſcribed in circles by the help of Iſoſceles triangles, whoſe angles at the baſe are multiples of thoſe at the top: and figures whoſe number of ſides is odd, are inſcribed in a circle by the help of Iſoſceles triangles, whoſe angles at the baſe are multiples ſeſquialter of thoſe at the top.

As in the Iſoſceles triangle CAB if the angle A=3C =B, then will AB be the ſide of a Heptagon. If A=4C, then is AB the ſide of an Enneagon. But if A=$\frac{1}{2}$C, then is AB the ſide of a ſquare. And if A=2$\frac{1}{2}$ CAB will ſubtend the ſixth part of the circumference, and likewiſe if A=3$\frac{1}{2}$C then will AB be the ſide of an Octagon.

## P R O P. XII. Plate III. *Fig.* 1.

*About a circle given* F A B C D E, *to deſcribe a pentagon* HIKLG, *equilateral and equiangular.*

a 11. 4. (*a*) Inſcribe a pentagon ABCDE in the circle given; and from the center draw the right-lines FA, FB, FC, FD, FE; and to thoſe lines draw ſo many perpendiculars GAH, HBI, ICK, KDL, LEG, meeting in the points H, I, K, L, G, and the thing is done. For becauſe GA, GE from the ſame point G (*b*) touch the circle, (*c*) therefore is GA=GE, and (*d*) therefore the angle GFA =GFE, therefore the angle AFE = 2 GFA. After the ſame manner is the angle AFH = HFB, and conſequently the angle AFB = 2 AFH. (*e*) But the angle AFE=AFB, (*f*) therefore the angle GFA = AFH. But alſo the angle FAH (*g*) =FAG, and the ſide FA is common, (*h*) therefore HA=AG=GE=EL, &c. (*k*) Therefore HG, GL, LK, KI, IH, the ſides of the pentagon are equal, and ſo alſo are the angles, becauſe double of the equal angles AGF, AHF, therefore, &c.

b *cor.* 16. 3.
c 2. *cor.* 36.
d 8. 1.
e 27. 3.
f 7. *ax.*
g 12. *ax.*
h 26. 1.
k 2. *ax.*

*Coroll.*

After the ſame manner, if any equilateral and equiangled figure be deſcribed in a circle, and at the extreme points of the ſemi-diameters, drawn from the center to the angles, be drawn perpendicular lines to the ſaid diameters; I ſay, that theſe perpendiculars ſhall make another figure of as many equal ſides and equal angles, circumſcribed to the circle.

## PROP. XIII. Plate III. *Fig.* 2.

*In an equilateral and equiangular pentagon given* ABCDE *to inſcribe a circle* FGHK.

(*a*) Biſect two angles of the pentagon A and B, with the right-lines AI, BK, meeting in the point F. From F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle deſcribed from the center F through G will touch all the ſides of the pentagon.

a 9. 1.

Draw FC, FD, FE. Becauſe BA (*b*) =BC, and the ſide BF common, and the angle FBA(*c*) =FBC, (*d*) therefore is AF=FC, and the angle FAB = FCB, but the angle FAB (*e*) $=\frac{1}{2}$ BAE$=\frac{1}{2}$ BCD. Therefore the angle FCB$=\frac{1}{2}$ BCD. After the ſame manner are all the whole angles C, D, E biſected. Now whereas the angle FGB (*f*) =FHB, and the angle FBH=FBG, and the ſide FB is common, (*g*) therefore is FG=FH. In like manner are all the right-lines FH, FI, FK, FL, FG equal. Therefore a circle deſcribed from the center F, through G, paſſes through the points H, I, K, L, and (*h*) touches the ſide of the pentagon, becauſe the angles at thoſe points are right. *Which was to be done.*

b *hyp.*
c *conſtr.*
d 4. 1.
e *hyp.*
f 12. *ax.*
g 26. 1.
h *cor.* 16. 3.

*Coroll.*

Hence, if any two neareſt angles of an equilateral and equianglar figure are biſected, and from that point in which the lines meet that biſect the angles, be drawn right-lines to the remaining angles of the figure, all the angles of the figure ſhall be biſected.

*Schol.*

By the ſame method may a circle be inſcribed in any equilateral and equiangular figure.

### P R O P. XIV. Plate III. *Fig.* 1.

*About a pentagon given* ABCDE, *equilateral and equiangular, to deſcribe a circle* FABCDE.

Biſect any two angles of the pentagon with the right-lines AF, BF, meeting in the point F; the circle deſcribed from the center F through A ſhall be deſcribed about the pentagon.

a *cor.* 13. 4. For let FC, FD, FE be drawn. (*a*) Then the angles C,
b 6. 1. D, E are biſected; (*b*) and therefore FA, FB, FC, FD, FE are equal; therefore the circle deſcribed from the center F paſſes through A, B, C, D, E, all the angles of the pentagon. *Which was to be done.*

*Schol.*

By the ſame method is a circle deſcribed about any figure which is equilateral and equiangular.

### P R O P. XV. *Fig.* 3.

*In a circle given* GABCDEF *to inſcribe an Hexagon* (*or ſix-ſided figure*) ABCDEF *equilateral and equiangular.*

Draw the diameter AD; from the center D through the center G deſcribe a cirrle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. *Then I ſay it's done.*

a *cor.* 32. 1. For the angle CGD (*a*)=$\frac{1}{3}$ of 2 right-angles (*a*)=DGE
b 15. 1. (*b*)=AGF (*b*)=AGB. (*c*) Therefore BGC=$\frac{1}{3}$ of 2 right-
c *cor.* 13. 1. angles=FGE; therefore the (*d*) arches and (*e*) ſubtenſes
d 26. 3. AB, BC, CD, DE, EF, are equal. Therefore the hexa-
e 29. 3. gon is equilateral; but it is equiangled alſo, (*f*) becauſe
f 27. 3. all the angles of it ſtand upon equal arches.

*Corol.*

1. Hence, the ſide of an hexagon inſcribed in a circle is equal to the ſemidiameter,

2. Hereby

Plate II. Facing Pag. 58.

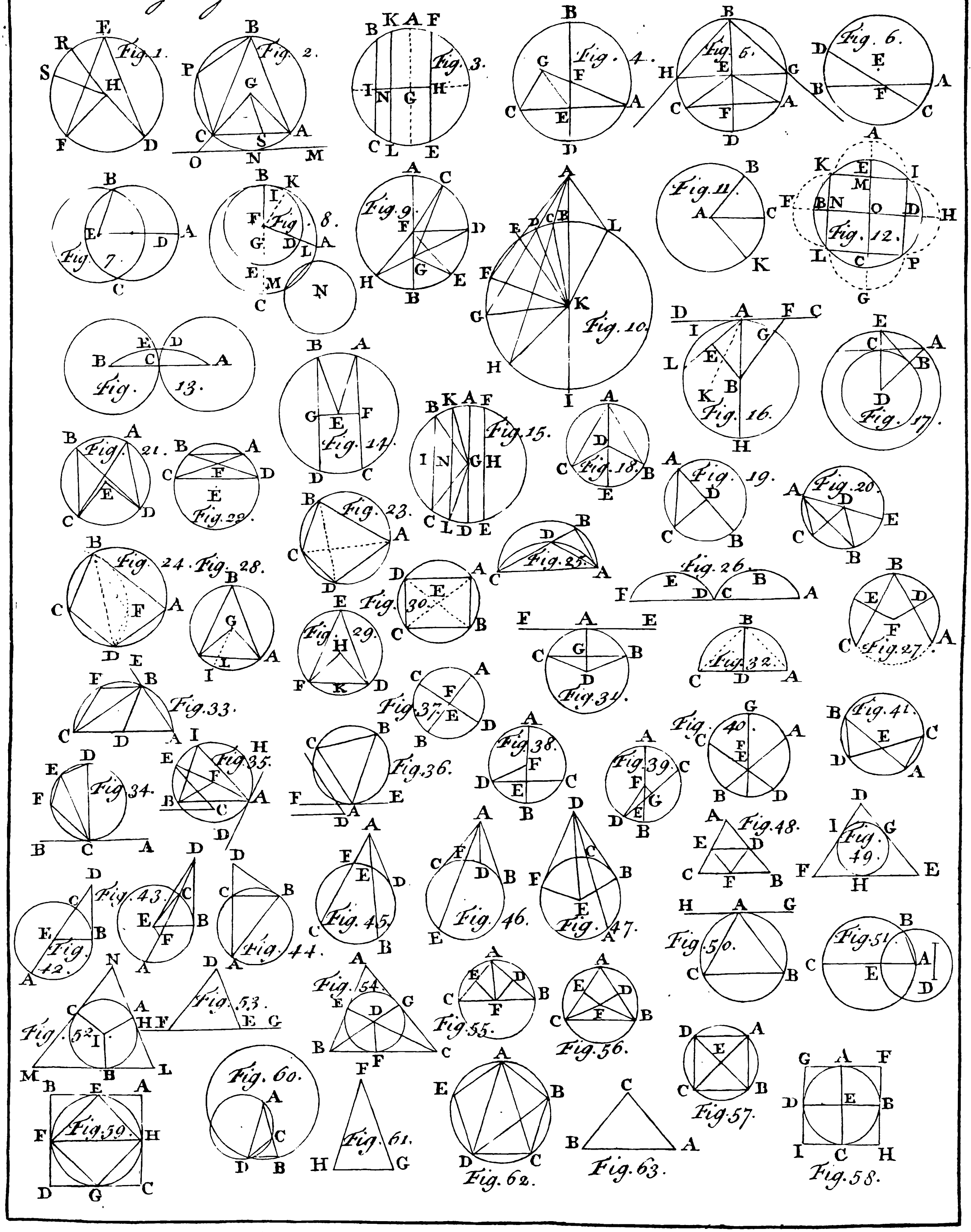
Fig. 1. R E S H F D
Fig. 2. B P G C S A O N M
Fig. 3. B K A F I N G H C L E
Fig. 4. B G F C E A D
Fig. 5. B H E G C F A D
Fig. 6. D E B F A C
Fig. 7. B E D A C
Fig. 8. B K I F G D L A E M C N
Fig. 9. A C F D H G E B
Fig. 10. A E D C B L F G K H I
Fig. 11. B A C K
Fig. 12. A K E I M F B N O D H L C P G
Fig. 13. E D B C A
Fig. 14. B A G E F D C
Fig. 15. B K A F I N G H C L D E
Fig. 16. D A F C I G L E K B H
Fig. 17. E C A B D
Fig. 18. A D C B E
Fig. 19. A D C B
Fig. 20. A D C E B
Fig. 21. B A E C D
Fig. 22. B A C F D E
Fig. 23. B A C D
Fig. 24. B C F A D
Fig. 25. D B C A
Fig. 26. F E D C B A
Fig. 27. B E D F C A
Fig. 28. B G A I L
Fig. 29. E H F K D
Fig. 30. D A E C B
Fig. 31. F A E C G B D
Fig. 32. B C D A
Fig. 33. F E B C D A
Fig. 34. E D F B C A
Fig. 35. I H E F B C A D
Fig. 36. C B F D A E
Fig. 37. C A F E D
Fig. 38. A F D E C B
Fig. 39. A F C G D E B
Fig. 40. G C F E A B D
Fig. 41. B C E D A
Fig. 42. D C E B A
Fig. 43. D C E B F A
Fig. 44. D C B
Fig. 45. A F E D C B
Fig. 46. A C F D B E
Fig. 47. D F C B E A
Fig. 48. A E D C F B
Fig. 49. D I G F E H
Fig. 50. H A G C B
Fig. 51. B C A E D
Fig. 52. N C A H I M B L
Fig. 53. D A F E G
Fig. 54. A E G D B F C
Fig. 55. A E D C F B
Fig. 56. A E D C F B
Fig. 57. D A E C B
Fig. 58. G A F D E B I C H
Fig. 59. B E A F H D G C
Fig. 60. A C D B
Fig. 61. F H G
Fig. 62. A E B D C
Fig. 63. C B A

2. Hereby an equilateral triangle ACE may very eaſily be deſcribed in a circle given.

*Schol. Probl.*

*To make a true hexagon upon a right-line given* CD. *Andr. Tacq.*
(*a*) Make an equilateral triangle CGD upon the line given CD; from the center G through C and D deſcribe a circle. That circle ſhall contain the hexagon made upon the given line CD. a 1. 1.

## PROP. XVI. Plate III. *Fig.* 4.

*In a circle given* AEBC, *to inſcribe a quindecagon* (*or fifteen-ſided figure*) *equilateral and equiangular.*

(*a*) Inſcribe an equilateral pentagon AEFGH in the circle given, and (*b*) alſo an equilateral triangle ABC, then I ſay BF is the ſide of the quindecagon required. a 11. 4. b 2. 4.

For the arch AB (*c*) is $\frac{1}{3}$ or $\frac{5}{15}$ of that periphery whereof AF is $\frac{2}{5}$ or $\frac{6}{15}$, therefore the remaining part BF is $\frac{1}{15}$ of the periphery; and therefore the quindecagon, whoſe ſide is BF, is equilateral; but it is equiangular alſo (*d*) becauſe all the angles ſtand on equal arches of a circle, whereof every one $\frac{13}{15}$ of the whole circumferecence. Therefore, &c. c *conſtr.* d 27. 3.

*Schol.*

| A circle is geometrically divided into parts | |
|---|---|
| | 4, 8, 16, &c. by 6, 4, and 9, 1. |
| | 3, 6, 12, &c. by 15, 4, and 9, 1. |
| | 5, 10, 20, &c. by 11, 4, and 9, 1. |
| | 15, 30, 60, &c. by 16, 4, and 9, 1. |

Any other way of dividing the circumference into any parts given is as yet unknown; wherefore in the conſtruction of ordinate figures, we are forced to have recourſe to mechanic artifices, concerning which you may conſult the writers of practical Geometry.

## *The End of the fourth* Book.

The

# The FIFTH BOOK OF EUCLIDE'S ELEMENTS.

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## *Definitions.*

I. A Part, is a magnitude of a magnitude, a leſs of a greater, when the leſs meaſureth the greater.

II. Multiple, is a greater magnitude in reſpect of a leſſer, when the leſſer meaſureth the greater.

III. Ratio, is the mutual habitude or reſpect of two magnitudes of the ſame kind each to other, according to quantity.

*In every ratio that quantity which is referr'd to another quantity, is called the antecedent of the ratio, and that to which the other is referr'd, is called the conſequent of the ratio, as in the ratio of* 6 *to* 4, 6 *is the antecedent and* 4 *the conſequent.*

Note, *The quantity of any ratio is known by dividing the antecedent by the conſequent; as the ratio of* 12 *to* 5 *is expreſſed by* $1\frac{2}{5}$; *or the quantity of the ratio of* A *to* B *is* $\frac{A}{B}$. *Wherefore, often for brevity ſake we denote the quantities of ratio's thus;* $\frac{A}{B} \sqsubset$, or $=$, or $\sqsupset \frac{C}{D}$. *that is, the ratio of* A *to* B *is greater, equal, or leſs than the ratio of* C *to* D. *Which muſt be well obſerved by thoſe who would underſtand this Book.*

*Concerning the divers ſpecies of ratio's, you may pleaſe to conſult interpreters.*

IV Proportion is a ſimiltude of ratio's.

*That which is here termed* proportion, *is more properly called* proportionality *or* analogy : *for proportion commonly denotes no more than the ratio betwixt two magnitudes.*

V. Thoſe numbers are ſaid to have a ratio betwixt them, which being multiplied may exceed one the other.

| | | |
|---|---|---|
| E, 12 | A, 4. B, 6. | G, 24. |
| F, 30 | C, 10 D, 15. | H, 60. |

VI. Magnitudes are ſaid to be in the ſame ratio, the firſt A, to the ſecond B, and the third C, to the fourth D, when the equimultiples E and F of the firſt A, and the third C, compared with the equimultiples G, H, of the ſecond B, and the fourth D, according to any multiplication whatſoever, either both together E, F are leſs than G, H both together, or equal taken together, or exceed one the other together, if thoſe be taken E, G, and F, H, which anſwer one to the other.

*The note hereof is* : : ; *as* A : B : : C : D. *That is, as* A *is to* B, *ſo is* C *to* D, *which ſignifies that* A *to* B, *and* C *to* D, *are in the ſame ratio. We ſometimes thus expreſs it* $\frac{A}{B} = \frac{C}{D}$, *that is,* A : B : : C : D.

VII. Magnitudes that have the ſame ratio (A : B : : C : D, are called proportional.

| | | |
|---|---|---|
| E, 30. | A, 6. B, 4. | G, 28. |
| F, 60. | C, 12. D, 9. | H, 63. |

VIII. When of equimultiples, E, the multiple of the firſt magnitude A, exceeds G, the multiple of the ſecond B, but F, the multiple of the third C, exceeds not H, the multiple of the fourth D, then the firſt A to the ſecond B, has a greater ratio than the third C to the fourth D.

*If* $\frac{A}{B} \sqsubset, \frac{C}{D}$, *it is not neceſſary from this definition, that* E *ſhould always exceed* G, *when* F *is leſs than* H ; *but it is granted that this may be.*

IX. Proportion conſiſts in three terms at leaſt. *Whereof the ſecond ſupplies the place of two.*

X. When three magnitudes A,B,C, are proportional, the firſt A is ſaid to have a duplicate ratio to the third C, of that it hath to the ſecond B : But when four magnitudes A, B, C, D are proportional, the firſt A is ſaid to have a triplicate ratio to the fourth D, of what it has to the ſecond B ; and ſo always in order one more, as the proportion ſhall be extended.

*Duplicate ratio is thus expreſſed* $\frac{A}{C} = \frac{A}{B}$ *twice, that is, the ratio of* A *to* C *is duplicate of the ratio of* A *to* B.

*Triplicate ratio is thus expreſſed;* $\frac{A}{D} = \frac{A}{B}$ *thrice; that is, the ratio of* A *to* D *is triplicate of the ratio of* A *to* B.

∺ *Denotes continued proportionals; as* A, B, C, D, ∺ *alſo* 2, 6, 18, 54, ∺ *are continual proportionals.*

XI. Homologous, or Magnitudes of like ratio, are antecedents to antecedents, and conſequents to conſequents; that is, *if* A : B : : C : D; A and C; as well as B and D are called homologous magnitudes.

XII. Alternate proportion is the comparing of antecedent to antecedent, and conſequent to conſequent. *As if* A : B : : C : D. *therefore alternately, or by permutation,* As A : C : : B : D. *by the* 16. *of* 5.

*In this definition, and the five following, names are given to the ſix ways of arguing which are often uſed by Mathematicians: the force of which inferences depends on the propoſitions of this Book, which are named in their explications.*

XIII. Inverſe ratio is when the conſequent is taken as the antecedent, and ſo compared to the antecedent as the conſequent; *as* A : B : : C : D; *therefore inverſly* B : A : : D : C. *by cor.* 4. 5.

XIV. Compounded ratio is when the antecedent and conſequent taken both as one, are compared to the conſequent it ſelf. *As* A : B : : C : D; *therefore by compoſition* A+B : B : : C+D : D, *by* 18. 5.

XV. Divided ratio is when the exceſs wherein the antecedent exceedeth the conſequent, is compared to the conſequent. *As* A : B : : C : D; *therefore by diviſion,* A—B : B : : C - D : D, *by* 17. 5.

XVI. Converſe ratio is when the antecedent is compared to the exceſs wherein the antecedent exceeds the conſequent. *As* A : B : : C : D; *therefore by converſe ratio* A : A—B : : C : C—D, *by the coroll. of the* 19 *of the* 5.

XVII. Proportion of equality is where there are taken more magnitudes than two in one order, and alſo as many magnitudes in another order, comparing two to two being in the ſame ratio; it follows that as in the firſt order of magnitudes, the firſt is ~~to~~ the laſt, ~~ſo in~~ the ſecond order of magnitudes, is the firſt to the laſt. Or otherwiſe: it is compariſon of the extremes together, the mean magnitudes being omitted.

*Thus let* A, B, C, *be three magnitudes and* D, E, F, *three others, and taking them two by two, let them be in the ſame proportion, that is, let* A : B : : D : E, *and* B : C : : E : F; *now if it be inferr'd that* A, *the firſt of the firſt order*

*der, is to* C *the laſt, as* D *the firſt of the ſecond order, is to* F *the laſt, this form of arguing is ſaid to be* ex æquo, *or from equality.*

XVIII. Ordinate proportion is, when antecedent is to conſequent, as antecedent to conſequent, and as the conſequent is to any other, ſo is the conſequent to any other. *As when* A : B : : D : E. *alſo* B : C : : E : F. *and then it ſhall be* A : C : : D : F, *by the* 22. *of the* 5.

XIX. Perturbate proportion is, when three magnitudes being put, and others alſo, which are equal to theſe in multitude, as in the firſt magnitudes the antecedent is to the conſequent, ſo in the ſecond magnitudes is the antecedent to the conſequent: and as in the firſt magnitudes the conſequent is to any other, ſo in the ſecond magnitudes is any other, to the antecedent. *Thus if* A, B, C, *and* E, F, G, *are two ſets of magnitudes, if* A *the firſt of the firſt ſet, is to* B *the ſecond, as* F *the ſecond of the ſecand ſet, is to* G *the laſt; and alſo if* B *the ſecond of the firſt ſet is to* C *the laſt, as* E *the firſt of the ſecond ſet is to* F *the ſecond, ſuch is called pertubate proportion, and by the* 23. 5. A : C : : E : G.

XX. Any number of magnitudes being put; the proportion of the firſt to the laſt is compounded out of the proportions of the firſt to the ſecond, the ſecond to the third, and the third to the fourth, *&c.* to the laſt.

Let there be any number of magnitudes A, B, C, D, by this definition

$$\frac{A}{D} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}.$$

*Axiom.*

Equimultiples of the ſame, or of equal magnitudes are equal to each other.

### PROP. I. Plate III. *Fig.* 5.

*If there be any number of magnitndes* AB, CD, *equimultiples to a like number of magnitudes* E, F, *each to each; what ever multiple one magnitude* AB *is of one* E, *the ſame multiple is all the magnitudes* AB+CD *to all the other magnitudes* E+F.

Let AG, GH, HB, the parts of the quantity AB, be equal to E, and alſo let CI, IK, KD, the parts of the quantity CD, be equal to F. The Number of theſe are put equal to thoſe. Now whereas AG+CI (*a*) = E+F; a 2. *ax.*

a 2. ax. F; (*a*) and $GH+IK=E+F$; (*a*) and $HB+KD=E+F$, it is evident that $AB+CD$ doth ſo often contain $E+F$ as one AB contains E, *Which was to be done.*

## PROP. II. Plate III. *Fig.* 6.

*If the firſt* AB *be the ſame multiple of the ſecond* C, *as the third* DE *is of the fourth* F, *and if the fifth* BG *be the ſame multiple of the ſecond* C, *as the ſixth* EH *is of the fourth* F; *then ſhall the firſt and fifth taken together* (AG) *be the ſame multiple of the ſecond* C, *as the third and ſixth taken together* (DH) *is of the fourth* F.

The number of parts in AB equal each to C is put equal to the number of parts in DE, whereof each part is equal to F. Likewiſe the number of parts BG is put equal to the number of parts in EH. Therefore the number of parts in $AB+BG$ is equal to the number of parts in $DE+EH$. a 2. ax. (*a*) That is, the whole line AG is the ſame multiple of C, as the whole line DH is of F. *Which was to be demonſtrated.*

## PROP. III. *Fig.* 7.

*If the firſt* A *be the ſame multiple of the ſecond* B, *as the third* C *is of the fourth* D, *and there be taken* EI, FM *equimultiples of the firſt and third, then will each of the magnitudes taken be equimultiples, the one* EI *of the ſecond* B, *the the other* FM *of the fourth* D.

Let EG, GH, HI, the parts of the multiple EI, be equal to A; alſo let FK, KL, LM, the parts of the multiple FM be equal to C; a hyp. (*a*) the number of theſe is equal to the number of thoſe. Moreover A (that is) EG or GH or HI, is put the ſame multiple of B, as C, or FK, &c. is of D. b 2. 5. (*b*) Therefore $EG+GH$ is the ſame multiple of the ſecond B, as $FK+KL$ is of the fourth D. c 2. 5. (*c*) By the ſame way of arguing is EI ($EH+HI$) the ſame multiple of B, as FM ($FL+LM$) is of D. *Which was to be demonſtrated.*

## PROP. IV. *Fig.* 8.

*If the firſt* A *have the ſame ratio to the ſecond* B, *as the third* C *to the fourth* D; *then alſo* E *and* F *the equimultiples of the firſt* A *and the third* C, *ſhall have the ſame ratio to* G *and* H, *the equimultiples of the ſecond* B *and the fourth* D, *according to any multiplication, if ſo taken as they anſwer each to other* (E : G : : F : H.)

Take I and K equimultiples of E and F; and alſo L and M equimultiples of G and H. (*a*) Then is I the ſame multiple of A, as K is of C; (*a*) and alſo L is the ſame multiple of B, as M of D. Therefore whereas it is A : B (*b*) : : C : D; according to the ſixth definition, if I be ⊏, =, ⊐ L, then conſequently after the ſame manner is K ⊏, =, ⊐ M, Therefore when I and K are taken the ſame multiples of E and F, as L and M of G, and H, then will it be by the ſeventh definition E : G : : F : H. *Which was to be demonſtrated.*

a 3. 5.
b *hyp.*

*Coroll.*

*From hence is generally demonſtrated the proof of inverſe ratio.*

For becauſe A : B : : C : D, therefore if E ⊏, =, ⊐ G, then is (*c*) likewiſe F ⊏, =, ⊐ H; therefore it is evident, that if G ⊏, =, ⊐ E, then is H ⊏, =, ⊐ F; (*d*) therefore B : A : : D : C. *Which was to be demonſtrated.*

c 2. *def.* 5.
a 6. *def.* 5.

### PROP. V. Plate III. *Fig.* 9.

*If a magnitude* AB *be the ſame multiple of a magnitude* CD, *as a part taken from the one* AE *is of a part taken from the other* CF; *the reſidue of the one* EB, *ſhall be the ſame multiple of the reſidue of the other* FD *as the whole* AB *is of the whole* CD.

Take another GA, which ſhall be the ſame multiple of FD, the reſidue, as AB is of the whole CD, or as the part taken away AE, is of the part taken away CF. (*a*) Therefore the whole GA + AE is the ſame multiple of the whole CF + FD, as the one AE is of the one CF, that is, as AB is of CD; therefore GE (*b*) = AB; and (*c*) ſo AE, which is common, being taken away, there remains GA = EB. Therefore, *&c.*

a 1. 5.
b 6. *ax.*
c 3. *ax.*

### PROP. VI. *Fig.* 10.

*If two magnitudes* AB, CD, *are equimultiples of two magnitudes* E, F; *and ſome magnitudes* AG *and* CH *equimultiples of the ſame* E, F, *be taken away; then the reſidues* GB, HD, *are either equal to theſe magnitudes* E, F, *or elſe equimultiples of them.*

For becauſe the number of parts in AB, whereof each is equal to E, is put equal to the number of parts in CD, whereof each is equal to F; and alſo the number of parts in AG equal to the number of parts in CH; If from one you take AG, and from the other CH, (*a*) there remains the number of parts in the remainder GB equal to the number of parts in HD; therefore if GB be once E, then is HB once C; if GB be many times E, then is HD ſo many times C. *Which was to be demonſtrated.*

a 3. *ax.*

### PROP. VII. Plate III. *Fig.* 11.

*Equal magnitudes* A *and* B *have the ſame proportion or ratio to the ſame magnitude* C. *And one the ſame magnitude* C *hath the ſame ratio to equal magnitudes* A *and* B.

Take D and E equimultiples of the equal magnitudes A and B, and F any multiple of C; then is D (*a*) $=$ E. Wherefore if D $\sqsubset$, $=$, $\sqsupset$ F, then alſo E will be $\sqsubset$, $=$, $\sqsupset$ F; (*b*) therefore A : C : : B : C; and (*c*) by inverſion C : A : : (*c*) C : B. *Which was to be demonſtrated.*

a 6. *ax.*
b 6. *def.* 5.
c *cor.* 4. 5.

### *Schol.*

If inſtead of the multiple F, two equimultiples be taken, it will be the ſame way proved that equal magnitudes have the ſame ratio to any other magnitudes that are equal between themſelves.

### PROP. VIII. *Fig.* 12.

*Of unequal magnitudes* AB, AC, *the greater* AB *hath a greater ratio to the ſame third line* D, *than the leſſer* AC; *and the ſame third line* D *hath a greater ratio to the leſſer* AC, *than to a greater* AB.

Take EF, EG equimultiples of the ſaid AB, AC, ſo that EH, a multiple of D, may be greater than EG, but leſſer than EF, (which will eaſily happen, if both EG and GF be taken greater than D.) It is manieſt from 8 *def.* 5. that $\frac{AB}{D} \sqsubset \frac{AC}{D}$ and $\frac{D}{AB} \sqsupset \frac{D}{AC}$. *Which was to be demonſtrated.*

### PROP. IX. *Fig.* 13.

*Magnitudes, which to one and the ſame magnitude have the ſame ratio, are equal the one to the other. And if a magnitude have the ſame ratio to other magnitudes, thoſe magnitudes are equal one to the other.*

1. *Hyp.* If A : C : : B : C ; I ſay that A = B. For let A be greater or leſs than C ; (*a*) then is $\frac{A}{C} \sqsubset$ or $\sqsupset \frac{B}{C}$. a 8. 5.
*Which is contrary to the Hypotheſis.*

2. *Hyp.* If C : B : : C : A. I ſay that A = B. For let A be $\sqsubset$ B, (*b*) then $\frac{C}{B} \sqsubset \frac{C}{A}$. *Which is againſt the Hypotheſis.* b 8. 5.

## PROP. X. Plate III. *Fig.* 14.

*Of magnitudes having ratio to the ſame magnitude, that which has the greater ratio, is the greater magnitude ; and that magnitude to which the ſame bears a greater ratio, is the leſſer magnitude.*

1. *Hyp.* If $\frac{A}{C} \sqsubset \frac{B}{C}$. I ſay that A $\sqsubset$ B.

For if it be ſaid that A = B. (*a*) then A : C : : B : C. *Which* a 7. 5.
*is contrary to the Hypotheſis.* If A $\sqsupset$ B, (*b*) then is b 8. 5.
$\frac{A}{C} \sqsupset \frac{B}{C}$ *Which is alſo againſt the Hypotheſis.*

2. *Hyp.* If $\frac{C}{B} \sqsubset \frac{C}{A}$. I ſay that B $\sqsupset$ A ; for if you ſay B = A, it's againſt the Hypotheſis, for it will c 7. 5.
(*c*) follow that C : B : : C : A. If you ſay B $\sqsubset$ A, (*d*) then d 8. 5.
is $\frac{C}{A} \sqsubset \frac{C}{B}$. *Which is alſo aginſt the Hypotheſis.*

## PROP. XI. *Fig.* 15, 16, 17.

*Proportions which are one and the ſame to any third, are alſo the ſame one to another.*

Let A : B : : E : F, and C : D : : E : F. I ſay that A : B : : C : D. Take G, H, I, equimultiples of A, C, E ; and K, L, M, equimultiples of B, D, F. Now (*a*) be- a *hyp.*
cauſe A : B : : E : F ; if G $\sqsubset$, =, $\sqsupset$ K, (*b*) then after the ſame manner I $\sqsubset$, =, $\sqsupset$ M. And likewiſe (*a*) becauſe E : F : : C : D, if I $\sqsubset$, =, $\sqsupset$ M, (*b*) then is b 6. *def.* 5.
H likewiſe $\sqsubset$, =, $\sqsupset$ L (*c*) wherefore A : B : : C : D. c 6. *def* 5.
*Which was to be demonſtrated.*

### *Schol.*

Proportions that are one and the ſame to the ſame proportions, are the ſame betwixt themſelves.

## PROP. XII. Plate III. *Fig.* 15, 16, 17.

*If any number of magnitudes* A, B, C, D, E *and* F *be proportional; as one of the antecedents* A, *is to one of the conſequents* B, *ſo are all the antecedents* A, C, E, *to all the conſequents*, B, D, F.

Take G, H, I, equimultiples of the antecedents, and
K, L, M, of the conſequents. Becauſe that one G is
a 1. 5. the ſame multiple of one A, (*a*) as all G, H, I, are of
all A, C, E; and becauſe one K is the ſame multiple
of one B, as all K, L, M, are of all B, D, F: More-
b *hyp.* over becauſe A : B (*b*) : : C : D (*b*) : : E : F, if G be ⊏,
=, or ⊐ K, then will H likewiſe be ⊏, =, ⊐ L,
and I ⊏, =, ⊐ M; and ſo if G ⊏, =, ⊐ K, in
like manner will G+H+I be ⊏, =, ⊐ K+L+M,
c 6. *def.* 5. (*c*) wherefore A : B : : A+C+E : B+D+F. *Which was to be demonſtrated.*

### *Corol.*

Hence, if like proportionals be added to like proportionals, the wholes ſhall be proportional.

## PROP. XIII. *Fig.* 15, 16 17.

*If the firſt* A *have the ſame ratio to the ſecond* B, *that the third* C *hath to the fourth* D, *and if the third* C *have a greater proportion to the fourth* D, *than the fifth* E *to the ſixth* F; *then alſo ſhall the firſt* A *have a greater proportion to the ſecond* B, *than the fifth* E *to the ſixth* F.

Take G, H, I, equimultiples of A, C, E, and K, L,
M, equimultiples of B, D, F. Now becauſe that A :
a 6. *def.* 5. B : : C : D, if H ⊏ L, (*a*) then is G ⊏ K; but becauſe
b 8. *def.* 5. $\frac{C}{D} \sqsubset \frac{E}{F}$, (*b*) it may be that H ⊏ L, and yet I not ⊏
c 8 *def.* 5. M. (*c*) Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$. *Which was to be demon.*

*Schol.*

But if $\frac{C}{D} \sqsupset \frac{E}{F}$, then alſo is $\frac{A}{B} \sqsupset \frac{E}{F}$. Alſo, if $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if $\frac{A}{B} \sqsupset \frac{C}{D} \sqsupset \frac{E}{F}$, then is $\frac{A}{B} \sqsupset \frac{E}{F}$.

## PROP. XIV. Plate III. *Fig.* 18.

*If the firſt* A *have the ſame ratio to the ſecond* B, *that the third* C *hath to the fourth* D; *and if the firſt* A *be greater than the third* C; *then ſhall the ſecond* B *be greater than the fourth* D. *But if the firſt* A *be equal to the third* C, *then the ſecond* B *ſhall be equal to the fourth* D; *but if* A *be leſs, then is* B *alſo leſs.*

Let A $\sqsubset$ C; (*a*) then $\frac{A}{B} \sqsubset \frac{C}{B}$, (*b*) but $\frac{A}{B} = \frac{C}{D}$; (*c*) therefore $\frac{C}{D} \sqsubset \frac{C}{B}$; (*c*) therefore B $\sqsubset$ D. By the like way of argument, if A $\sqsupset$ C, (*d*) then is B $\sqsupset$ D. But if A be put equal to C, then C : B : : (*e*) A : B (*f*) : : C : D. (*g*) Therefore B $=$ D. *Which was to be demonſtrated.*

a 8. 5.
b *hyp.*
c 13. 5.
d 10. 5.
e 7. 5.
f *hyp.*
g 11. 5. & 9. 5.

*Schol.*

By an argument *à fortiori*, if $\frac{A}{B} \sqsupset \frac{C}{D}$, and A $\sqsubset$ C, then is B $\sqsubset$ D. Likewiſe if A$=$B, then is C$=$D, and if A $\sqsubset$, or $\sqsupset$ B, then alſo is C $\sqsubset$ or $\sqsupset$ D.

## PROP. XV. *Fig.* 19.

*Parts* C *and* F *are in the ſame ratio, with their like multiples* AB *and* DE, *if taken correſpondently.* (AB : DE : : C : F.)

Let AG, GB, parts of the multiple AB, be equal to C; and let DH, HE, parts of the multiple DE, be equal to F. (*a*) The number of theſe parts is equal to the number

a *hyp.*

b 7. 5. ber of thoſe. Therefore whereas (*b*) AG : C : : DH : F,
c 12. 5. and GB : C : : HE : F ; therefore is (*c*) AG+GB (AB)
: DH+HE (DE) : : C : F. *Which was to be demonſtrated.*

### P R O P. XVI. Plate III. *Fig.* 20.

*If four magnitudes* A, B, C, D, *are proportional, they alſo ſhall be alternately proportional* (A : C : : B : D)

a 15. 5. Take E and F equimultiples of A and B; take alſo G
b *hyp.* and H equimultiples of C and D. Therefore E : F (*a*)
c 11. 5. & : : A : B (*b*) : : C : D (*a*) : : G : H. Wherefore if E
14. 5. $\sqsubset$, $=$, $\sqsupset$ (*c*) G, then likewiſe is F $\sqsubset$, $=$, $\sqsupset$ H. (*d*)
d 6. *def.* 5. Therefore A : C : : B : D. *Which was to be demonſtrated.*

*Schol.*

Alternate ratio has place only when the quantities are of the ſame kind. For heterogeneous quantities are not compared together.

### P R O P. XVII. *Fig.* 21.

*If magnitudes compounded are proportional* (AB : CB : : DF : FE.) *they ſhall be proportional alſo when divided.* (AC : CB : : DF : FE.)

Take GH, HL, IK, KM, in order equimultiples of
AC, CB, DF, FE; and alſo LN, MO, equimultiples
a 1. 5 of CB, FE. The whole GL is (*a*) the ſame multiple of
b *conſtr.* the whole AB, as one GH is of one AC; (*b*) that is, as
c 1. 5. IK of DF, (*c*) or as the whole IM of the whole DE.
d 2. 5. Alſo HN (HL+LN) is (*d*) the ſame multiple of CB,
as KO, (KM+MO) is of FE. Therefore, whereas by
Hyp. AB : BC : : DE : EF, if GL be $\sqsubset$, $=$, $\sqsupset$ HN,
e 6. *def.* 5. then likewiſe (*e*) will IM $\sqsubset$, $=$, $\sqsupset$ KO. Take from
theſe HL, KM, that are equal; and if the remainder
f 5. *ax.* GH be $\sqsubset$, $=$, $\sqsupset$ LN, (*f*) then will IK $\sqsubset$, $=$, $\sqsupset$
g 6. *def.* 5. MO; (*g*) whence AC : CB : : DF, FE. *Which was to be demonſtrated.*

### P R O P. XVIII. *Fig.* 22.

*If magnitudes divided are proportional* (AB : BC : : DE : EF,) *the ſame alſo being compounded ſhall be proportional* (AC : CB : : DF : FE.)

a 17. 5. For if it can be, let AC : CB : : DF : FG $\sqsupset$ FE. (*a*)
b *hyp.* & Then by diviſion will AB : BC : : DG : GF; (*b*) that is,
11. 5. DG : GF : : DE : EF; and ſince DG $\sqsubset$ DE, (*c*) there-
c 14. 5. fore is GF $\sqsubset$ EF. *Which is abſurd.* The like abſurdity
d 9. *ax.* will follow if it be ſaid AC : CB : : DF : GF $\sqsubset$ FE (*d*).

P R O P.

### PROP. XIX. Plate III. *Fig.* 23.

*If the whole* AB *be to the whole* DE *as the part taken away* AC *is to the part taken away* DF, *then ſhall the reſidue* CB *be to the reſidue* FE, *as the whole* AB *is to the whole* DE.

Becauſe (*a*) AB : DE : : AC : DF (*b*) therefore by permutation AB : AC : : DE : DF, (*c*) and thence by diviſion AC : CB : : DF : FE; (*b*) wherefore again by permutation AC : DF : : CB : FE (*d*) that is, AB : DE : : CB : FE. *Which was to be demonſtrated.*

a *hyp.*
b 16. 5.
c 17. 5.
d *hyp.* & 11. 5.

*Coroll.*

Hence, If like proportionals be ſubtracted from like proportionals, the remainders ſhall be proportional.

2. *Hence is converſe ratio demonſtrated.*

Let AB : CB : : DE : FE. I ſay that AB : AC : : DE : DF. For by (*a*) permutation AB : DE : : CB : FE, (*b*) therefore AB : DE : : AC : DF, whence again by permutation AB : AC : : DE : DF. *Which was to be demonſtrated.*

a 16. 5.
b 19. 5.

### PROP. XX. *Fig.* 24.

*If there are three magnitudes* A, B, C, *and others* D, E, F, *equal to thoſe in number, which being taken two and two in each order are in the ſame ratio,* (A : B : : D : E; and B : C : : E : F,) *and if* ex æquo *the firſt* A *be greater than the third* C; *then ſhall the fourth* D *be greater than the ſixth* F. *But if the firſt* A *be equal to the third* C, *then the fourth* D *ſhall be equal to the ſixth* F; *and if* A *be leſs than* C, *then ſhall* D *be leſs than* F.

1. *Hyp.* If A $\sqsubset$ C. Becauſe (*a*) E : F : : B : C, by (*b*) inverſion it ſhall be F : E : : C : B. (*c*) But $\frac{C}{B} \sqsupset \frac{A}{B}$, (*d*) therefore $\frac{F}{E} \sqsupset \frac{A}{B}$ or $\frac{D}{E}$, (*e*) therefore D $\sqsubset$ F. *Which was to be demonſtrated.*

a *hyp.*
b *cor.* 4. 5.
c *hyp.* and 8. 5.
d *ſchol.* 13. 5
e 10. 5.

2. *Hyp.* By the ſame way of arguing, if A $\sqsupset$ C, it will appear that D $\sqsupset$ F.

3. *Hyp.* If A = C. Becauſe F : E : : C : B : : (*f*) A : B : : D : E, (*g*) therefore is D = F. *Which was to be dem.*

f 7. 5.
g 11. 5. & 9. 5.

### PROP. XXI. *Fig.* 24.

*If there are three magnitudes* A, B, C, *and others alſo* D, E, F, *equal to them in number, which taken two and two* *are in the ſame ratio ; and their proportion pertubate* (A : B : : E : F, *and* B : C : : D : E.) *and if* ex æquo *the firſt* A *be greater than the third* C, *then is the fourth* D *greater than the ſixth* F ; *but if the firſt be equal to the third, then is the fourth equal to the ſixth ; if leſs, ſo is the other likewiſe.*

a *hyp.* 1. *Hyp.* If A $\sqsubset$ C ; then becauſe (*a*) D : E : : B : C,
b 8. 5. therefore inverſely E : D : : C : B, but $\frac{C}{B}$ (*b*) $\sqsupset \frac{A}{B}$ ;
c *ſch.* 13. 5. (*c*) therefore $\frac{E}{D} \sqsupset \frac{A}{B}$, that is, than $\frac{E}{F}$, (*d*) therefore
d 10. 5. D $\sqsubset$ F.

2. *Hyp.* By the like argument, if A $\sqsupset$ C, then is D $\sqsupset$ F.

e 7. 5. 3. *Hyp.* If A = C ; then becauſe E : D : : (*e*) C : B
f *hyp.* : : (*e*) A : B : : (*f*) E : F, (*g*) therefore is D = F.
g 9. 5. *Which was to be demonſtrated.*

## PROP. XXII. Plate III. *Fig.* 25.

*If there be any number of magnitudes* A, B, C, *and others equal to them in number* D, E, F, *which taken two and two are in the ſame ratio* (A : B : : D : E *and* B : C : : E : F) *they ſhall be in the ſame ratio alſo by equality* (A : C : : D : F.)

Take G, H, equimultiples of A, D; and I, K, of B, E ; and alſo L, M, of E, F.

a *hyp.* Becauſe (*a*) A : B : : D : E, (*b*) therefore G : I : : H :
b 4. 5. K ; and in like manner I : L : : K : M. therefore if G
c 20. 5. $\sqsubset$, =, $\sqsupset$ L, (*c*) then is H $\sqsubset$, =, $\sqsupset$ M ; (*d*) there-
d 6. *def.* 5. fore A : C : : D : F. By the ſame way of demonſtration if further C : N : : F : O, then by equality A : N : : D : O. *Which was to be demonſtrated.*

## PROP. XXIII. *Fig.* 26.

*If there are three magnitudes* A, B, C, *and others* D, E, F, *equal to them in number, which taken two and two are in the ſame ratio, and their proportion perturbate* (A : B : : E : F, *and* B : C : : D : E) *they ſhall be in the ſame ratio alſo by equality* (A : C : : D : F.)

Take G, H, I, equimultiples of A, B, D ; and alſo K, L, M, equimultiples of C, E, F. Then G : H : : (*a*)
a 15. 5. A : B : : (*b*) E : F (*a*) : : L : M. Moreover becauſe (*b*)
b *hyp.* B : C : : D : E, thence is (*c*) H : K : : I : L ; therefore
c 4. 5. G, H, K, and I, L, M, are as in 21. 5. Therefore if

G

G be ⊏, =, ⊐ K, then is likewiſe I ⊏, =, ⊐ M,
and ſo (*d*) conſequently A : C : : D : F. *Which was to* d 6. *def.* 5.
*be demonſtrated.*

*If there are more magnitudes than three, this way of demonſtration holds good in them alſo.*

*Coroll.*

From hence * it follows, that ratio's compounded of * 22 & 23.
the ſame ratio's, are among themſelves the ſame; as al- 5. *and* 20.
ſo that the ſame parts of the ſame ratio's, are among *def.* 1.
themſelves the ſame.

### PROP. XXIV. Plate III. *Fig.* 27.

*If the firſt magnitude* AB, *has the ſame ratio* ~~*to the*~~ *ſecond* C, *which the third* DE, *has to the fourth* F; *and if the fifth* BG *has the ſame ratio to the ſecond* C, *which the ſixth* EH *has to the fourth* F; *then ſhall the firſt compounded with the fifth* (AG) *have the ſame ratio to the ſecond* C, *which the third compounded with the ſixth* (DH) *has to the fourth* F.

For becauſe (*a*) AB : C : : DE : F, and by the Hyp. a *hyp.*
and inverſion C : BG : : F : EH; therefore by (*b*) equa- b 22. 5.
lity AB : BG : : DE : EH, whence by compounding, c *hyp.*
AG : BG : : DH : EH. Alſo (*c*) BG : C : : EH : F.
Therefore again by (*b*) equality AG : C : : DH : F.
*Which was to be demonſtrated.*

### PROP. XXV. *Fig.* 28.

*If four magnitudes are proportional* (AB : CD : : E : F) *the greateſt* AB *and the leaſt* F *ſhall be greater than the reſt* C, D, *and* E.

Make AG = E, and CH = F. Becauſe AB : CD : :
(*a*) E : F : : (*b*)AG : CH, (*c*) thence is AB : CD : : GB a *hyp.*
: HD; (*d*) but AB ⊏ CD, (*e*) therefore GB ⊏ HD. b 7. 5.
But AG+F=E+CH, therefore AG+F+GB ⊏ E c 19. 5.
+CH+HD; that is, AB+F ⊏ E+CD. *Which was* d *hyp.*
*to be demonſtrated.* e *ſch.* 14. 5.

Theſe propoſitions which follow are not *Euclide*'s, but taken out of other Authors, and are here ſubjoyned becauſe of their frequent uſe.

### PROP. XXVI. *Fig.* 29.

*If the firſt have a greater proportion to the ſecond, than the third to the fourth; then by converſion, the ſecond ſhall have a leſs proportion to the firſt, than the fourth to the third.*

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, I ſay that $\frac{B}{A} \sqsupset \frac{D}{C}$. For conceive $\frac{C}{D} = \frac{E}{B}$; (*a*) therefore $\frac{A}{B} \sqsubset \frac{E}{B}$; (*b*) whence $A \sqsubset E$; (*c*) there-

a 13. 5.
b 10. 5.
c 8. 5.
d *cor.* 4. 5.

fore $\frac{B}{A} \sqsupset \frac{B}{E}$, (*d*) or $\frac{D}{C}$. *Which was so be demonſtrated.*

## PROP. XXVII. Plate III. *Fig.* 29.

*If the firſt have a greater proportion to the ſecond, than the third to the fourth; then alternately the firſt ſhall have a greater proportion to the third, than the ſecond to the fourth.*

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, then I ſay $\frac{A}{C} \sqsubset \frac{B}{D}$. For conceive $\frac{E}{B} = \frac{C}{D}$, (*a*) therefore $A \sqsubset E$, (*b*) and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$.

a 10. 5.
b 1. 5.
c 16. 5.

(*c*) or $\frac{B}{D}$. *Which was to be demonſtrated.*

## PROP. XXVIII. *Fig.* 30.

*If the firſt have a greater proportion to the ſecond than the third to the fourth, then the firſt compounded with the ſecond ſhall have a greater proportion to the ſecond, than the third compounded with the fourth, to the fourth.*

Let $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. I ſay that $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. For conceive $\frac{GB}{BC} = \frac{DE}{EF}$, (*a*) therefore is, $AB \sqsubset GB$, add BC to each part, then (*b*) will $AC \sqsubset GC$; (*c*) therefore $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$, (*d*) that is, $\frac{DF}{EF}$. *Which was to be dem.*

a 10. 5.
b 4. *ax.*
c 8. 5.
d 18. 5.

## PROP. XXIX. *Fig.* 30.

*If the firſt compounded with the ſecond has a greater proportion to the ſecond, than the third compounded with the*

*the fourth hath to the fourth; then by diviſion the firſt ſhall have a greater proportion to the ſecond, than the third to the fourth.*

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$, then I ſay $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. For conceive $\frac{GC}{BC} = \frac{DF}{EF}$, (*a*) therefore AC $\sqsubset$ GC. Take a 10, 5.
away BC, which is common, there (*b*) remains AB $\sqsubset$ b 5. *ax.*
c 8. 4.
GB; (*c*) therefore $\frac{AB}{BC} \sqsubset \frac{GB}{BC}$ (*d*) or $\frac{DE}{EF}$. *Which* d 17. 5.
*was to be demonſtrated.*

## PROP. XXX, Plate III. *Fig.* 31.

*If the firſt compounded with the ſecond, has a greater proportion to the ſecond, than the third compounded with the fourth, hath to the fourth; then by converſe ratio ſhall the firſt compounded with the ſecond have a leſſer ratio to the firſt, than the third compounded with the fourth ſhall have to the third.*

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. Then I ſay that $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$.

For becauſe that $\frac{AC}{BC}$ (*a*) $\sqsubset \frac{DF}{EF}$, (*b*) therefore, by diviſion, a *hyp.*
b 29. 5.
$\frac{AB}{BC} \sqsubset \frac{DE}{EF}$, by converſion (*c*) therefore $\frac{BC}{AB} \sqsupset \frac{EF}{DE}$ c 26. 5.
and (*d*) therefore by compoſition $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. *Which* d 28. 5.
*was to be demonſtrated.*

## PROP. XXXI. *Fig.* 32.

*If there are three magnitudes* A, B, C, *and others alſo* D, E, F, *equal to them in number; and if there be a greater proportion of the firſt of the former to the ſecond, than there is of the firſt of the laſt to the ſecond* $\left(\frac{A}{B} \sqsubset \frac{D}{E};\right)$ *and there be alſo a greater proportion of the ſecond of the firſt magnitudes to their third, than there* *is of the ſecond of the laſt magnitudes to their third* $\left(\frac{B}{C} \sqsubset \frac{E}{F}.\right)$ *Then by equality alſo ſhall the ratio of the firſt of the former magnitudes to the third, be greater than the ratio of the firſt of the latter magnitudes, to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}.\right)$

a 10. 5. b 8. 5. Conceive $\frac{G}{C} = \frac{E}{F}$, (*a*) then is $B \sqsubset G$, and (*b*) therefore
c 13. 5. $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{E}$, (*c*) then
d 10. 5. $\frac{H}{G} \sqsupset \frac{A}{B}$, therefore much more $\frac{H}{G} \sqsupset \frac{A}{G}$, (*d*) wherefore
e 8. 2. f 22. 5. $A \sqsubset H$, (*e*) and conſequently $\frac{A}{C} \sqsubset \frac{H}{C}$, (*f*) or $\frac{D}{F}$. *W.W.D.*

## PROP. XXXII. Plate III. *Fig.* 32.

*If there be three magnitudes* A, B, C, *and others* D, E, F, *equal to them in number; and there be a greater proportion of the firſt of the former magnitudes to the ſecond, than there is of the ſecond of the latter to the third* $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ *and alſo the ratio of the ſecond of the former to the third be greater than the ratio of the firſt of the latter to the ſecond* $\left(\frac{B}{C} \sqsubset \frac{D}{E}.\right)$ *then by equality alſo ſhall the proportion of the firſt of the former to the third, be greater than that of the firſt of the latter to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}.\right)$

a 10. 5. Suppoſe $\frac{G}{C} = \frac{D}{E}$ then is (*a*) $B \sqsubset G$, and therefore
b 8. 5. c ſchol. 13. 5. (*b*) $\frac{A}{G} \sqsubset \frac{A}{B}$. Again, Suppoſe $\frac{H}{G} = \frac{E}{F}$; therefore is (*c*)
$\frac{H}{G} \sqsupset \frac{A}{G}$, and conſequently (*a*) $A \sqsubset H$, and thence (*b*)

$\frac{A}{C} \sqsubset \frac{H}{C}$ (*d*) or $\frac{D}{F}$. *Which was to be demonſtrated.* d 13. 5.

## PROP. XXXIII. Plate III. *Fig.* 33.

*If the proportion of the whole* AB *to the whole* CD *be greater than the proportion of the part taken away* AE *to the part taken away* CF; *then ſhall alſo the ratio of the remainder* EB *to the remainder* FD *be greater than that of the whole* AB *to the whole* CD.

Becauſe that $\frac{AB}{CD}$ (*a*) $\sqsubset \frac{AE}{CF}$, (*b*) therefore by permutation $\frac{AB}{AE} \sqsubset \frac{CD}{CF}$, (*c*) therefore by converſe ratio $\frac{AB}{EB} \sqsupset \frac{CD}{FD}$, and by permutation again $\frac{AB}{CD} \sqsupset \frac{EB}{FD}$. *Which was to be demonſtrated.* a *hyp.* b 27. 1. c 30. 5.

## PROP. XXXIV.

*If there be any number of magnitudes, and others alſo equal to them in number; and the proportion of the firſt of the former to the firſt of the latter be greater than that of the ſecond to the ſecond, and that greater than the proportion of the third to the third, and ſo forward: all the former magnitudes together ſhall have a greater ratio to all the latter together, than all the former, leaving out the firſt, ſhall have to the latter, leaving out the firſt; but leſs than that of the firſt of the former to the firſt of the latter; and laſtly, greater than that of the laſt of the former to the laſt of the latter.*

You may pleaſe to conſult Interpreters for the demonſtration hereof, we having for brevity ſake omitted it, and becauſe 'tis of no uſe in theſe Elements.

## *The End of the fifth* BOOK.

The

# The SIXTH BOOK OF EUCLIDE's ELEMENTS.

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## *Definitions.*

Plate III. Fig. 33. I. LIke right-lined figures (ABC, DCE) are ſuch whoſe ſeveral angles are equal one to the other, and alſo their ſides about the equal angles, proportional.

*The Angle* B = DCE, *and* AB : BC : : DC : CE. *Alſo the angle* A = D, *and* BA : AC : : CD : DE. *Laſtly the angle* ACB = E, *and* BC : CA : : CE : ED.

Fig. 34. II. Reciprocal figures are (BD, BF) when in each of the figures there are terms both antecedent and conſequent (that is, AB : BG : : EB : BC.

Fig. 23. III. A right-line AB is ſaid to be cut according to extreme and mean proportion, when as the whole AB is to the greater ſegment AC, ſo is the greater ſegment AC to the leſs CB (AB : AC : : AC : CB.)

Fig. 33. IV. The altitude of any figure ABC, is a perpendicular line AG, drawn from the top A, to the baſe BC.

V. A ratio is ſaid to be compounded of ratio's, when the quantities of the ratio's, being multiplied into one another, produce a ratio, *As the ratio of* A *to* C *is compounded of the ratio's of* A *to* B *and* B *to* C. *For* $\frac{A}{B}\times\frac{B}{C}$

a 20. def. 5.
b 15. 5.

(a) $=\frac{AB}{BC}$ (b) $=\frac{A}{C}$.

### PROP. I. Plate III. Fig. 35.

*Triangles* ABC, ACD, *and parallelograms* BCAE, CDFA, *which have the ſame height, are to each other, as their baſes,* BC, CD.

(*a*) Take as many as you pleaſe, BG, GH, equal to a 3. 1.
BC, and alſo DI=CD, and join AG, AH, AI.
(*b*) The triangles ACB, ABG. AGH, are equal, and b 38. 1.
(*b*) alſo the triangle ACD=ADI. Therefore the triangle
ACH is the ſame multiple of the triangle ACB, as the
baſe HC is of the baſe BC; and the triangle AGI the
ſame multiple of the triangle ACD, as the baſe CI is of
CD. But if HC $\sqsubset$, $=$, $\sqsupset$ CI, (*c*) then is likewiſe the c *ſch*, 38. 1.
triangle AHC $\sqsubset$, $=$, $\sqsupset$ ACI; and (*d*) therefore BC : d 6. *def.* 5.
CD : : the triangle ABC : ACD : : (*e*) Pgr. CE : CF. e 41. 1. &
*Which was to be demonſtrated.* 15. 5.

### *Schol.* Plate III. *Fig.* 37.

Hence *triangles,* ABC, DEF, *and Pgrs.* AGBC, DEFH,
*whoſe baſes* BC, EF *are equal, are to each other as their*
*altitudes,* AI, DK.
(*a*) Take IL = CB, and KM = EF; and join LA, a 3. 1.
LG, MD, MH, then it is evident, that the triangle b 7. 5.
ABC : DEF : : (*b*) ALI : DKM : : (*c*) AI : DK : : (*d*) Pgr. c 1. 6.
AGBC : DEFH. *Which was to be demonſtrated.* d 41. 1. &
15. 5.

### PROP. II. *Fig.* 36.

*If to one ſide* BC *of a triangle* ABC, *be drawn a parallel right-line* DE, *the ſame ſhall cut the ſides of the triangle proportionally* (AB : BD : : AE : EC.) *And if the ſides of the triangle are cut proportionally* (AD : BD : : AE : EC) *then a right-line* DE, *joining the points of a ſection* DE, *ſhall be parallel to* BC, *the other ſide of the triangle. Draw* CD *and* BE.

1. *Hyp.* Becauſe the triangle DEB (*a*) =DEC, (*b*) there- a 37. 1.
fore ſhall be the triangle ADE : DBE : : ADE : ECD. b 7. 5.
But the triangle AED : DBE : : (*c*) AD : DB, and the c 1. 6.
triangle ADE : DEC : : AE : EC; (*d*) therefore AD : d 11. 5.
DB : : AE : EC. e 1. 6.
2. *Hyp.* Becauſe AD : DB : : AE : EC, (*e*) that is as f 9. 5.
the triangle ADE : DBE : : ADE : ECD; (*f*) therefore g 39. 1.
is the triangle DBE=ECD; and (*g*) therefore DE, BC
are parallels. *Which was to be demonſtrated.*

### *Schol.*

If there are drawn ſeveral lines DE, FG parallel to one ſide BC of a triangle, all the ſegments of the ſides ſhall be proportional.

For DF : FA (*a*) : : EG : GA ; and compounding
a 2. 6. and inverting, FA : DA : : GA : EA ; (*a*) and DA : DB
: : EA : EC ; therefore by equality DF : DB : : EG : EC.
*Which was to be demonſtrated.*

### *Coroll.*

If DF : DB : : EG : EC ; (*a*) then BC, DE, FG, ſhall be parallels.

## PROP. III. Plate III. *Fig.* 38.

*If an angle* BAC *of a triangle* BCA *be biſected, and the right-line* AD, *that biſects the angle, cut the baſe alſo; then ſhall the ſegments of the baſe have the ſame ratio that the other ſides of the triangle have,* (BD : DC : : AB : AC.) *And if the ſegments of the baſe have the ſame ratio, that the other ſides of the triangle have* (BD : DC : : AB : AC) *then a right-line* AD *drawn from the top* A *to the ſection* D, *ſhall biſect that angle* BAC *of the triangle.*

Produce BA, and make AE=AC, and join CE.
a 5. 1. 1. *Hyp.* Becauſe AE=AC, therefore is the angle
b 32. 1. ACE (*a*)=E (*b*)= half BAC (*c*) = DAC ; (*d*) there-
c *hyp.* fore DA, CE are parallels. (*e*) Wherefore BA : AE (AC)
d 27. 1. : : BD : DC,
e 2. 6. 2. *Hyp.* Becauſe BA : AC (AE) : : BD : DC, (*f*) there-
f 2. 6. fore are DA, CE parallels ; and (*g*) therefore is the angle
g 29. 1. BAD=E ; and the angle DAC (*g*)=ACE (*h*) = E, (*k*)
h 5. 1 therefore the angle BAD=DAC. Wherefore the angle
k 1. *ax.* BAC is biſected. *Which was to be demonſtrated.*

## PROP. IV. *Fig.* 39.

*Of equiangular triangles* ABC, DCE, *the ſides are proportional which are about the equal angles,* B, DCE, (AB : BC : : DC : CE, &c.) *And the ſides* AB, DC, &c. *which are ſubtended under the equal angles* ACB, E, &c. *are homologous, or of like ratio.*

Set the ſide BC in a direct line to the ſide CE, and
a 32. 1. & produce BA and ED till they (*a*) meet in F.
13. *ax.* Becauſe the angle B (*b*) =ECD, (*c*) therefore BF, CD
b *hyp.* are parallel: Alſo becauſe the angle BCA (*b*) = CED,
c 28. 1. (*e*) therefore are CA, EF parallel. Therefore the figure
d 34. 1. CAFD is a Pgr. (*d*) therefore AF=CD, and AC = (*d*)
e 2. 6. FD. Whence it is evident, that AB : AF (CD) : : (*e*)

BC : CE. (*f*) By permutation therefore AB : BC : : CD :
CE, alſo BC : CE : : FD (AC) : DE. (*f*) And thence by f 16. 5.
permutation BC : AC : : CE : DE. (*g*) Wherefore alſo g 22. 5.
by equality AB : AC : : CD : DE. Therefore, *&c.*

*Coroll.*

Hence AB : DC : : BC : CE : : AC : DE.

*Schol.*

Hence, if in a triangle FBE there be drawn AC, a parallel to one ſide FE, the triangle ABC ſhall be ſimilar or like to the whole FBE.

### PROP. V. Plate III. *Fig.* 40, 41.

*If two triangles* ABC, DEF, *have their ſides proportional* (AB : BC : : DE : EF, *and* AC : BC : : DF : EE, *and alſo* AB : AC : : DE : DF.) *thoſe triangles are equiangular, and thoſe angles equal, under which are ſubtended the homologous ſides.*

At the ſide EF (*a*) make the angle FEG=B, and the a 23. 1.
angle EFG=C; (*b*) whence the angle G=A. Therefore b 32. 1.
GE : EF (*c*) : : AB : BC : : (*d*) DE : EF. (*e*) And c 4. 6.
therefore GE=DE. Likewiſe GF : FE (*c*) : : AC : CB d *hyp.*
: : (*d*) DF : FE; (*e*) therefore GF=DF. Therefore the e 11. 5. &
triangles DEF, GEF, are mutually equilateral. (*f*) 9. 5.
Therefore the angle D=G=A, and the angle FED (*f*) f 8. 1.
=FEG=B, and (*g*) conſequently the angle DFE=C. g 32. 1.
Therefore, *&c.*

### PROP. VI. *Fig.* 40, 41.

*If two triangles* ABC, DEF *have one angle* B *equal to one angle* DEF, *and the ſides about the equal angles* B, DEF *proportional* (AB : BC : : DE : EF) *then thoſe triangles* ABC, DEF, *are equiangular, and have thoſe angles equal, under which are ſubtended the homologous ſides.*

At the ſide EF make the angle FEG = B, and the a 32. 1.
angle EFG=C; (*a*) then will the angle G=A. Therefore b 4. 6.
GE : EF : : (*b*) AB : BC : : (*c*) DE : EF, (*d*) and c *hyp.*
therefore DE=GE. But the angle DEF (*e*) = B (*f*) d 9. 5.
= GEF; therefore the angle D (*g*) = G = A, (*h*) and e *hyp.*
conſequently the angle EFD = C. *Which was to be* f *conſtr.*
*demonſtrated.* g 4. 1.
h 32. 1.

## PROP. VII. Plate III. *Fig.* 43, 44.

*If two triangles* ABC, DEF *have one angle* A *equal to one angle* D, *and the ſides about the other angles* ABC, E, *proportional* (AB : BC : : DE : EF) *and if they have the remaining angles* C, F, *either both leſs or both greater than a right-angle; then ſhall the triangles* ABC, DEF, *be equiangular, and have thoſe angles equal about which the proportional ſides are.*

For, if it can be, let the angle ABC $\sqsubset$ E, and make the angle ABG = E. Therefore, whereas the
a *hyp.* angle A (*a*) = D, (*b*) thence is the angle AGB = F.
b 32. 1. Therefore AB : BG (*c*) : : DE : EF : : (*d*) AB : BC, (*e*)
c 4. 6. therefore BG = BC; (*f*) therefore the angle BGC =
d *hyp.* BCG. (*g*) Therefore BGC, or C, is leſs than a right-
e 9. 5. angle, and (*h*) conſequently AGB or F is greater than a
f 5. 1. right: Therefore the angles C and F are not of the ſame
g *cor.* 17. 1. ſpecies or kind, which is againſt the Hypotheſis.
h *cor.* 13. 1.

## PROP. VIII. *Fig.* 42.

*If a line* AD *be drawn from the right-angle* A, *of a right-angled triangle* ABC, *perpendicular to the baſe* BC; *then the triangles* ADB, ADC, *on each ſide the perpendicular, are ſimilar both to the whole* ABC, *and to one another.*

a *hyp.* For becauſe BAC, ADB are (*a*) right-angles, (*b*) and
b 12. *ax.* ſo equal, and B common; the triangles BAC, ADB, (*c*)
c 32. 1. & are like. By the ſame way of arguing BAC, ADC,
4. 6. are like; (*d*) whence alſo ADB, ADC will be like.
d *vid.* 21. 6. *Which was to be demonſtrated.*

### *Coroll.*

e 1. *def.* 6. Hence, 1. BD : DA (*e*) : : DA : DC.
2. BC : AC : : AC : DC, and CB : BA : : BA : BD.

## PROP. IX. *Fig.* 45.

*From a right-line given* AB *to cut off any part required, as one third* (AG.)

From the point A draw an infinite line AC any wiſe,
a 3. 1. in which (*a*) take any three equal parts AD, DE, EF;
b 31. 1. join FB, to which, from D, (*b*) draw the parallel DG,
and the thing is done.

For GB : AG : : (*c*) FD : AD; whence, by (*d*) compoſition, c 2. 6.
AB : AG : : AF : AD; therefore ſince AD= d 18. 5.
one third of AF, AG ſhall be = one third of AB.
*Which was to be done.*

### PROP. X. Plate III. *Fig.* 46.

*To divide a given undivided right-line,* AB (*in* F, G) *as another given right-line is divided* (*in* D, E.)

Let a right-line BC join the extremities of the line divided, and of the line not divided; and parallel to this,
from the points E, D, (*a*) draw EG, DF, meeting with a 31. 1.
the right-line which is to be cut in G and F; then the thing is done.

For let DH be (*a*) drawn parallel to AB. Then AD : b 2. 6.
DE : : (*b*) AF : FG, and DE : EC (*b*) : : DI : IH : : (*c*) c 34. 1. &
FG : GB. *Which was to be done.* 7. 5.

### *Schol. Fig.* 47.

*Hence we learn to cut a right-line given* AB, *into as many equal parts as we pleaſe* (*ſuppoſe* 5;) which will be more eaſily performed thus.

Draw an infinite line AD, and another BH parallel to it, and infinite alſo. Of theſe take equal parts, AR, RS, SV, VN; and BZ, ZX, XT, TL; in each line leſs parts by one, then are required in AB; then let the right-lines LR, TS, XV, ZN, be drawn; theſe lines ſo drawn ſhall cut the right-line given AB into five equal parts.

For RL, ST, VX, NZ, are (*a*) parallels; therefore a 33. 1.
whereas AR, RS, SV, VN are (*b*) equal; (*c*) thence AM, b *conſtr.*
MO, OP, PQ, are equal alſo. Likewiſe, becauſe that c 2. 6.
BZ = ZX, therefore is BQ=PQ, and therefore AB
is cut into five equal parts. *Which was to be done.*

### PROP. XI. *Fig.* 48.

*Two right-lines being given* AB, AD, *to find out a third in proportion to them* (DE.)

Join BD, and from AB, being produced, take BC= AD. Through C draw CE parallel to BD; with which let AD produced meet in E, then is DE the proportional required.

For AB : BC (AD) (*a*) : : AD : DE. *Which was to be done.* a 2. 6.

Or thus: make the angle BDA (*Fig.* 42.) right, and b 1. *cor.*8.6. alſo the angle BAC right, then (*b*) BD : DA : : DA : DC.

### P R O P. XII. Plate III. *Fig.* 49.

*Three right-lines being given,* DE, EF, DG, *to find out a fourth proportional,* GH.

Join E G, and through F draw F H parallel to E G; with which let DG, produced to H, meet. Then it is a 2. 6. evident that DE : EF (*a*) : : DG : GH. *Which was to be done.*

### P R O P. XIII. *Fig.* 50.

*Two right-lines being given* AE, EB, *to find a mean proportional,* EF.

Upon the whole line A B, as a diameter, deſcribe a ſemicircle A F B, and from E erect a perpendicular FE meeting with the periphery in F, then AE : EF : : EF : EB. For let AF and FB be drawn; a 31. 3. (*a*) then from the right-angle of the right-angled triangle AFB is drawn b *cor.* 8. 6. a right-line F E, perpendicular to the baſe. (*b*) Therefore AE : FE : : FE : EB. *Which was to be done.*

*Coroll.*

Hence, a right-line drawn in a circle from any point of a diameter, perpendicular to that diameter, and produced to the circumference, is a mean proportional betwixt the two ſegments of that diameter.

### P R O P. XIV. *Fig.* 34.

*Equal Parallelograms* BD, BF, *having one angle* ABC, *equal to one* EBG, *have the ſides which are about the equal angles reciprocal* (AB : BG : : EB : BC;) *and thoſe parallelograms* BD, BF, *which have one angle* ABC *equal to one* EBG, *and the ſides which are about the equal angles reciprocal, are equal.*

For let the ſides AB, BG, about the equal angles make a *ſch.* 15. 1. one right-line; (*a*) wherefore EB, BC, ſhall do the ſame. b 1. 6. Let FG, DC, be produced till they meet.

c 7. 5. d 1. 6. e 11. 5. 1. *Hyp.* AB : BG (*b*) : : BD : BH : : (*c*) BF : BH : : (*d*) BE : BC, (*e*) therefore, &c.

2. *Hyp*. BD : BH : : (*f*) AB : BG : : (*g*) BE : BC : : (*h*) BF : BH. (*k*) Therefore the Pgr. BD = BF. *Which was to be demonſtrated.*

f 1. 6.
g *hyp.*
h 1. 6.
k 11. *and* 9. 5.

### PROP. XV. Plate III. *Fig.* 51.

*Equal triangles having one angle* ABC, *equal to one* DBE, *their ſides which are about the equal angles are reciprocal* (AB : BE : : DB : BC.) *And thoſe triangles that have one angle* ABC *equal to one* DBE, *and have alſo the ſides that are about the equal angles reciprocal* (AB : BE : : DB : BC) *are equal.*

Let the ſides CB, BD, which are about the equal angles be ſet in a ſtrait-line; (*a*) therefore ABE is a right-line. Let CE be drawn.

1. *Hyp*. AB : BE : : (*b*) the triangle ABC : CBE (*c*) : : triangle DBE : CBE : : (*d*) DB : BC; (*e*) therefore, &c.

2. *Hyp*. The triangle ABC : CBE : : (*f*) AB : BE : : (*g*) DB : BC (*h*) : : the triangle DBE : CBE. (*k*) Therefore the triangle ABC = DBE. *Which was to be demonſtrated.*

a *ſch.* 15. 1.
b 1. 6.
c 7. 5.
d 1. 6.
e 11. 5.
f 1. 6.
g *hyp.*
h 1. 6.
k 11. *and* 9. 5.

### PROP. XVI. *Fig.* 52.

*If four right-lines are proportional* (AB : FG : : EF : CB) *the rectangle* AC, *comprehended under the extremes* AB, CB, *is equal to the rectangle* EG, *comprehended under the means* FG, EF. *And if the rectangle* AC, *comprehended under the extremes* AB, CB, *be equal to the rectangle* EG, *comprehended under the means* FG, EF, *then are the four right-lines proportional* (AB : FG : : EF. CB.)

1. *Hyp*. The angles B and F are right, and (*a*) conſequently equal, and by hypotheſis AB : FG : : EF : CB, (*b*) therefore the rectangle AC = EG.

2. *Hyp*. The rectangle AC (*c*) = EG, and the angle B = F; (*d*) therefore AB : FG : : EF : CB. *Which was to be demonſtrated.*

a 12. *ax.*
b 14. 6.
c *hyp.*
d 14. 6.

### *Coroll.*

Hence it is eaſy to apply a rectangle given EG to a right-line given AB; (*viz.*) (*e*) by making AB : EF : : FG : BC.

[e 12. 6.]

## PROP. XVII. Plate III. *Fig.* 52.

*If three right-lines are proportional* (AB : EF : : EF : CB) *the rectangle* AC, *made under the extremes* AB, CB, *is equal to the ſquare* EG, *made of the middle* EF. *And if the rectangle* A C, *comprehended under the extremes* A B, C B, *be equal to the ſquare* EG, *made of the middle* EF, *then the three lines are proportional,* (AB : EF : : EF : CB.)

Take FG=EF.

a *hyp.* 1. *Hyp.* AB : EF : : (*a*) EF (FG) : CB, therefore the
b 16. 6. rectangle AC (*b*) = EG (*c*) = EFq.
c 29. *def.* 1. 2. *Hyp.* The rectangle AC (*d*) = to the ſquare EG =
d *hyp.* EFq ; (*e*) therefore AB : EF : : FG (EF) : BC. *Which*
e 16. 6. *was to be demonſtrated.*

*Coroll.*

Let A×B = Cq, then A : C : : C : B.

## PROP. XVIII. *Fig.* 53.

*Upon a right-line given* AB, *to deſcribe a right-lined figure* AGHB, *ſimilar and alike ſituate to a right-lined figure given* CEFD.

Reſolve the right-lined figure given into triangles ; (*a*)
a 23. 1. Make the angle ABH = D, (*a*) and the angle BAH =
DCF, (*a*) and the angle AHG = CFE, (*a*) and the angle
H A G = F C E, then A G H B, ſhall be the right-lined
figure ſought,

For the angle B (*b*) = D, and the angle BAH (*b*) =
b *conſtr.* DCF, (*c*) wherefore the angle AHB = CFD ; (*b*) alſo
c 32. 1. the angle HAG=FCE, and the angle AHG (*b*) =
CFE ; (*c*) wherefore the angle G=E, and the whole
angle GAB (*d*) = ECD, and the whole angle GAB (*d*)
d 2. *ax.* = EFD. The Polygons therefore are mutually equian-
e 4. 6. gular. Moreover becauſe the triangles are equiangular,
therefore AB : BH (*e*) : : CD : DF ; and AG : GH (*e*)
: : CE : EF, Likewiſe AG : AH : : (*e*) CE : CF, and
f 22. 5. AH : AB : : CF : CD. (*f*) From whence by equality
AG : AB : : CE : CD. After the ſame manner GH :
g 1. *def.* 6. HB : : EF : FD. (*g*) Therefore the Polygons ABHG,
CDFE are ſimilar and alike ſituate. *Which was to be done.*

## PROP. XIX. *Fig.* 54.

*Like triangles* ABC, DEF, *are in duplicate ratio of their homologous ſides,* BC, EF.

(*a*) Let there be made BC : EF : : EF : BG, and let AG be drawn. Becauſe that AB : DE (*b*) : : BC : EF (*c*) : : EF : BG, and the angle B = E, (*d*) therefore is the triangle ABG = DEF. But the triangle ABC : ABG : : (*e*) BC : BG, and (*f*) $\frac{BC}{BG} = \frac{BC}{EF}$ twice; therefore $\frac{ABC}{ABG}$ that is, $\frac{ABC}{DEF}$ (*g*) $= \frac{BC}{EF}$ twice. *Which was to be demonſtrated.*

a 11. 6.
b *cor.* 4. 6.
c *conſtr.*
d 15. 6.
e 1. 6.
f 10. *def.* 5.
g 11. 5.

*Coroll.*

Hence, If three right-lines (BC, EF, BG) are proportional, then as the firſt is to the third, ſo is a triangle made upon the firſt BC, to a triangle ſimilar and alike deſcribed upon the ſecond EF; or ſo is a triangle deſcribed upon the ſecond EF, to a triangle ſimilar and alike deſcribed upon the third.

## PROP. XX. Plate III. *Fig.* 56, 57.

*Like Polygons* ABCDE, FGHIK, *are divided into equal triangles* ABC, FGH, *and* ACD, FHI, *and* ADE, FIK; *both equal in number and homologous to the wholes* (ABC : FGH : : ABCDE : FGHIK : : ACD : FHI : : ADE : FIK.) *And the Polygons* ABCDE, FGHIK, *have a duplicate ratio one to the other of what one homologous ſide* BC *hath to the other homologous ſide* GH.

1 For the angle B (*a*) = G, and AB : BC (*a*) : : FG : GH. (*b*) Therefore the triangles ABC, FGH, are equiangular. After the ſame manner are the triangles AED, FKI ſimilar. Since therefore the angle BCA (*b*) = GHF, and the angle ADE (*b*) = FIK, and the whole angles BCD, GHI, and the whole angles CDE, HIK are (*c*) equal, there remains the angle ACD (*d*) = FHI, and the angle ADC = FIH; (*e*) from whence alſo the angle CAD = HFI, therefore the triangles ACD, FHI are ſimilar. Therefore, &c.

a *hyp.*
b 6. 6.
c *hyp.*
d 3. *ax.*
e 32. 1.

2. Becauſe the triangles BCA, GHF are like, (*f*) therefore, $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the ſame reaſon is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; laſtly $\frac{DEA}{IKF} = \frac{DE}{IK}$ twice. Now whereas that BC : GH (*g*) : : CD : HI (*g*) : : DE : IK; (*h*) therefore is the triangle BCA : GHF : : CAD : HFI : : DEA : IKF (*k*) : : the polygon ABCDE : FGHIK : : BC : GH twice.

f 19. 6.
g *hyp.* & 16. 5.
h *cor.* 23. 5.
k 12. 5.

### *Coroll.*

1. Hence if there are three right-lines proportional, then as the firſt is to the third, ſo is a polygon made upon the firſt to a polygon made on the ſecond ſimilar and alike deſcribed; or ſo is a polygon made upon the ſecond, to a polygon made on the third ſimilar and alike deſcribed.

*Hence we have a method of inlarging or diminiſhing any right-lined figure in a ratio given*; For if you would make a pentagon quintuple of that pentagon whereof CD is the ſide, then betwixt AB and 5 AB find out a * 18. 6. mean proportional, * upon this raiſe a pentagon like to that given, and it ſhall be quintuple of the pentagon given.

2. Hence alſo, If the homologous ſides of like figures be known, then will the proportion of the figures be evident, *viz.* by finding out a third porportional.

## PROP. XXI. Plate III. *Fig.* 58.

*Right-lined figures* ABC, DIE, *which are ſimilar to the ſame right-lined figure* HFG, *are alſo ſimilar one to the other.*

For the angle A (*a*) = H (*a*) = D; and the angle C (*a*) a 1. *def.* 5. = G (*a*) = E; and the angle B (*a*) = F (*a*) = I. Alſo (*a*) AB : AC : : HF : HG : : DI : DE; and (*a*) AC : CB : : HG : GF : : DE : EI. And AB : BC : : HF : FG : : DI : IE. Therefore (*a*) ABC, DIE, are ſimilar. *Which was to be demonſtrated.*

## PROP. XXII. *Fig.* 59.

*If four right-lines are proportional* (AB : CD : : EF : GH) *the right-lined figures alſo deſcribed upon them being ſimilar and alike ſituate, ſhall be proportional* (ABI : CDK : : EM : GO.) *And if the right-lined figures deſcribed upon the lines, ſimilar and alike ſituate, be proportional* (ABI : CDK : : EM : GO) *then the right-lines alſo ſhall be proportional* (AB : CD : : EF : GH.)

a 19. 6. 1. *Hyp.* $\frac{ABI}{CDK}$ (*a*) = $\frac{AB}{CD}$ twice = (*a*) $\frac{EF}{GH}$ twice (*a*) = $\frac{EM}{GO}$ (*b*) therefore ABI : CDK : : EM : GO.

b *hyp.*
c 20. 6.
d *cor.* 23. 5

2. *Hyp.* $\frac{AB}{CD}$ twice (*a*) = $\frac{ABI}{CDK}$ (*b*) = $\frac{EM}{GO}$ = (*c*) $\frac{EF}{GH}$. twice. Therefore (*d*) AB : CD : : EF : GH. *Which was to be demonſtrated.*

*Schol.*

### *Schol.*

*Hence is deduced the manner and reaſon of multiplying ſurd quantities*, ex. g. Let $\sqrt{5}$ be to be multiplied into $\sqrt{3}$. I ſay that the product will be $\sqrt{15}$. For by the definition of multiplication it ought to be, as 1 : $\sqrt{3}$ :: $\sqrt{5}$, to the product. Therefore by this q. 1 : q. $\sqrt{3}$ :: q $\sqrt{5}$ : q. of the product. That is, 1 : 3 :: 5, to the ſquare of the product, therefore the ſquare of the product is 15. Wherefore $\sqrt{15}$ is the product of $\sqrt{3}$ into $\sqrt{5}$. *Which was to be demonſtrated.*

### *THEOREM.* Plate III. *Fig.* 50.

*If a right-line* AB *be cut any-wiſe in* E, *the rectangle* *Pet. Herig.*
*comprehended under the parts* AE, EB, *is a mean proportional betwixt their ſquares. Like-wiſe the rectangle comprehended under the whole* AB, *and one part* AE, *or* EB *is a mean proportional betwixt the ſquare of the whole* AB *and the ſquare of the ſaid part*, AE, *or* EB,

Upon the diameter A B deſcribe a ſemicircle ; from E erect a perpenpicular EF, meeting with the periphery in F, join AF, BF.

It's evident that AE : EF (*a*) : : EF : EB, (*b*) therefore a *cor.* 8. 6.
AEq : EFq. :: EFq : EBq. (*c*) that is, AEq : AEB : : b 22. 6.
AEB : EBq. *Which was to be demonſtrated.* c 17. 6.
Moreover BA : AF : : (*d*) AF : AE, (*e*) therefore BAq. d *cor.* 8. 6.
AFq : : AFq. : AEq. (*f*) that is, BAq : BAE : : BAE : e 22. 6.
AEq. After the ſame manner ABq. : ABE : : ABE : f 17. 6.
BEq. *Which was to be demonſtrated.*

Or thus: ſuppoſe Z = A + F. It is manifeſt that Aq.
: AF : : (*a*) A : F : : (*a*) AF : Fq. alſo Zq. : ZA : : (*a*) a 1. 6.
Z ; A : : ZA : Aq, and Zq : ZF : : (*a*) Z : F : : ZF : Fq.

### PROP. XXIII. *Fig.* 34.

*Equiangular parallelograms* AC, BF, *have the ratio one to the other, which is compounded of their ſides.* $\left(\frac{AC}{BF}=\frac{AB}{BG}+\frac{BC}{BE}\right)$

Let the ſides about the equal angles, B be (*a*) ſet in a a *ſch.* 19.
direct line, and let the Pgr. BH be compleated. Then is
the ratio of $\frac{AC}{BF}$ (*b*) $=\frac{AC}{BH}+\frac{BH}{BF}$ (*c*) $=\frac{AB}{BG}+\frac{BC}{BE}$ *Which* b 20 *def.* 5.
*was to be demonſtrated.* c 1. 6.

*Corell.*

*Coroll.* Plate III. *Fig.* 61.

*Andr.* Hence, *and from* 34. 1. *it appears,* 1. *That triangles*
*Tacq.* 15. 5 *which have one angle equal* (*as at* C) *have a ratio compounded of the ratio's of the right-lines,* (A *to* B, *and* C *to* F,) *containing the equal angle.*

* 35. 1. 2. *That all rectangles, and * conſequently all parallelograms, have their ratio one to the other compounded of the ratio's of baſe to baſe, and altitude to altitude.* After the like manner you may argue in triangles.

3. *From hence is apparent how to give the proportion of triangles and parallelograms.*

Let there be two Pgrs. X and Z, whoſe baſes are AC, CB, and altitudes CL, CF. Make CL : CF : : CB : O,
* 14 .6. and 1. 6. * then will it be X : Z : : AC :O.

## PROP. XXIV. *Fig.* 55.

*In every parallelogram* ABCD, *the parallelograms* EG, HF, *which are about the diameter* AC, *are like to the whole, and alſo one to the other.*

For the Pgrs. EG, HF, have each of them one an-
a 29. 1. gle common with the whole ; (*a*) therefore they are equiangular to the whole, and alſo one to the other. Alſo the triangles ABC, AEI, IHC (*a*) and the triangles ADC, AGI, IFC are equiangular mutually ; (*b*)
b 4. 6. therefore AE : EI : : AB : BC, and (*b*) AE : AI : : AB :
c 22. 5. AC, and (*b*) AI : AG : : AC : : AD, (*c*) Therefore by e-
d 1. *def.* 6. quality, AE : AG : : AB : AD. (*d*) Therefore the Pgrs. EG, BD are ſimilar, After the ſame manner are HF, BD ſimilar alſo. Therefore, &c.

## PROP. XXV. *Fig.* 62.

*Unto the right-lined figure given* ABEDC, *to deſcribe another figure* P, *ſimilar and alike ſituate, which alſo ſhall be equal to another right-lined figure given* F.

a 45. 1. (*a*) Make the rectangle AL = ABEDC ; (*b*) alſo upon
b 44. 1. BL make the rectangle BM = F ; betwixt AB und BH
c 13. 6. (*c*) find out a mean proportional NO ; (*e*) upon NO (*d*)
d 18. 6. make the polygon P ſimilar to the right-lined figure
e *cor.* 20. 6. given ABEDC. I ſay the polygon P ſo made, ſhall be
f 1. 6. equal to F, that was given.
g 14. 5. For ABEDC (AL) : P : : AB : BH : : (*f*) AL : BM.
h *conſtr.* Therefore P (*g*) = BM (*h*) = F. *Which was to be done.*

PROP.

## PROP. XXVI. Plate III. *Fig.* 60.

*If from the parallelogram* ABCD, *be taken away another parallelogram* AGFE, *ſimilar to the whole, and alike ſituate, having alſo an augle* EAG *common with it; then is that parallelogram about the ſame diagonal* AC *with the whole.*

If you deny AC to be the common diagonal, then let
AHC be it, cutting EF in H, and let HI be drawn paral-
lel to AE. Then are the Pgrs. EI, DB, (*a*) ſimilar; (*b*) a 24. 6.
therefore AE : EH : : AD : DC : : (*c*) AE : EF, and (*d*) b 1. *def.*6.
conſequently EH = EF. (*f*) *Which is abſurd.* c *hyp.*
d 9. 5.
f 9. *ax.*

## PROP. XXVII. Plate IV. *Fig.* 1.

*Of all parallelograms* AD, AG, *applied to the ſame right-line* A B, *and wanting in figure by the prallelograms* CE, KI, *ſimilar and alike ſituate to the Pgr.* AD, *which is deſcribed upon the half line, the greateſt is that* AD, *which is applied to the half line being like to the defect* KI.

For becauſe that GE (*a*) equal GC, if KI which is a 45. 1.
common be added, (*b*) then is KE = CI (*c*) = AM; b 2. *ax.*
add CG which is common, (*d*) then is AG = to the c 36. 1,
Gnomon MBL; (*e*) but the Gnomon MBL ⊐ CE (AD) d 2. *ax.*
Therefore AG ⊐ AD. *Which was to be demonſtrated.* e 9. *ax.*

## PROP. XXVIII. *Fig.* 2.

*To a right-line given* A B, *to apply a parallelogram* AP,
*equal to a right-lined figure given* C, *deficient by a parallelo-*
*gram* ZR, *which is ſimilar to another parallelogram given*
D, * *but it is neceſſary that the right-lined figure given* C, *to* * 27. 6.
*which the Pgr. to be applied* AP *muſt be equal, be not greater than the Pgr.* AF, *which is applied to half the line, ſince the defects both of* AF, *which is apply'd to half the line, and of* AP *the parallelogram to be applied, muſt be ſimilar.*

Biſect AB in E; upon EB (*a*) make the Pgr. EG like a 18. 6.
to the Pgr. D; and (*b*) let EG = C + I. (*c*) Make the b *ſch.* 45. 1
Pgr. NT = I, and like to the Pgr. given D, or EG; draw c 25. 6.
the diameter FB; Make FO = KN, and FQ = KT; thro' O and Q draw the parallels SR, QZ. Then is the Pgr. AP that which was ſought.

d *conſt.* & For the Pgrs. D, EG, OQ, NT, ZR, are all (*d*) ſimilar
24. 6. one to the other, and the Pgr. EG = (*e*) NT + C = (*e*)
e *conſtr.* OQ + C; (*f*) wherefore C = to the Gnomon OBQ (*g*)
f 3. *ax.* = AO + PG = (*h*) AO + EP = AP. *Which was to be done.*
g 2. *ax.*
h 43. 1.

## PROP. XXIX. Plate IV. *Fig* 3.

*Upon a right-line given* AB, *to apply a parallelogram* AN *equal to a right-lined figure given* C, *exceeding by a Pgr.* OP, *which ſhall be like to another Pgr. given* D.

a 18. 6. Biſect AB in E. Upon EB (*a*) make a Pgr. EG like to
b 25. 6. the given one D, and (*b*) let the Pgr. HK = EG + C, and
c 3. 1. like to the given one D, or to EG. Make FEL = (*c*)
IH; and (*c*) FGM = IK. Thro' L, M, draw the parallels MN and RN; and AR parallel to NM. Produce ABP, GBO; draw the diameter FBN. Then is AN the parallelograms required.

d *conſtr.* For the Pgrs. D, HK, LM, EG, are (*d*) ſimilar; (*e*)
e 24. 6. therefore the Pgr. OP is ſimilar to the Pgr. LM, or D.
f *conſtr.* Alſo LM (*f*) = HK (*f*) = EG + C. (*g*) Therefore C
g 3. *ax,* = to the Gnomon ENG. But AL (*h*) = LB (*k*) =
h 36. 1. BM; (*l*) therefore C = AN. *Which was to be done.*
k 43. 1.
l 2. *and* 1. *ax.*

## PROP. XXX. *Fig.* 5.

*To cut a finite right-line given* AB, *according to extreme and mean ratio* (AB : AG : : AG : GB.)

a 11. 2. (*a*) Cut AB in G, in ſuch wiſe that AB×BG = AGq. (*b*)
b 17. 6. Then BA : AG : : AG : GB. *Which was to be done.*

## PROP. XXXI. *Fig.* 6.

*In right-angled triangles* BAC, *any figure* BF *deſcribed upon the ſide* BC, *ſubtending the right-angle* BAC, *is equal to the figures* BG, AL, *which are ſimilar and alike ſituate to the former* BF, *and deſcribed upon the ſides* BA, AC, *containing the right-angle.*

From the right-angle BAC let fall the perpendicular
a *cor.* 8. 6. AD. Becauſe DC : CA : : (*a*) CA : CB, therefore AL :
b *cor.* 20. 6. BF : : DC : CB. Alſo, becauſe DB : BA : : (*a*) BA :
c 24. 5. BC, (*b*) therefore BG : BF : : DB : BC; (*c*) therefore
d *ſch.* 14. 5. AL + BG : BF : : DC + DB (BC) : BC. (*d*) Therefore
AL + BG = BF. *Which was to be demonſtrated.*

e 22. 6. Or thus: BG : BF : : (*e*) BAq : BCq : And (*e*) AL :
f 24. 5. BF : : ACq : BCq, (*f*) therefore BG + AL : BF : :
g *ſch.* 14. 5. BAq + ACq : BCq. (*g*) Therefore whereas BAq + ACq
h 47. 1. = (*h*) BCq; (*h*) thence is BG + AL = BF. *Which was to be demonſtrated.*

*Coroll.*

Plate III. *facing Pag.* 92.

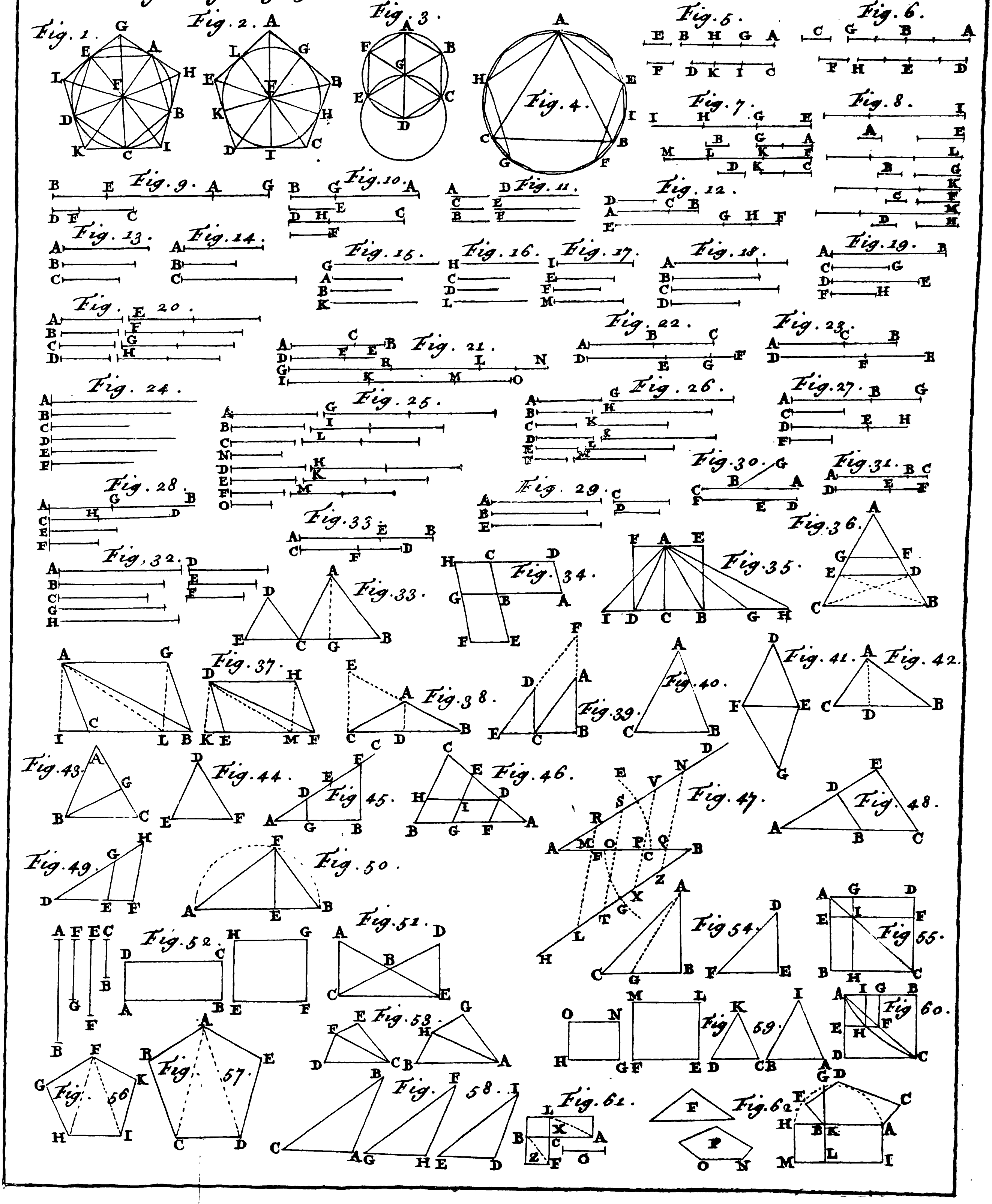
Fig. 1. G E A L H F D B K C I
Fig. 2. A L G E B F K H D I C
Fig. 3. A F B G E C D
Fig. 4. A H E I C B G F
Fig. 5. E B H G A F D K I C
Fig. 6. C G B A F H E D
Fig. 7. I H G E B G A M L K F D K C
Fig. 8. I A E L B G K C F M D H
Fig. 9. B E A G D F C
Fig. 10. B G A D H E C F
Fig. 11. A D C E B F
Fig. 12. D C B A G H F E
Fig. 13. A B C
Fig. 14. A B C
Fig. 15. G A B K
Fig. 16. H C D L
Fig. 17. I E F M
Fig. 18. A B C D
Fig. 19. A B C G D E F H
Fig. 20. A E B F C G D H
Fig. 21. A C B D F E G R L N I K M O
Fig. 22. A B C D E G F
Fig. 23. A C B D F E
Fig. 24. A B C D E F
Fig. 25. A G B I C L N D H K E F M O
Fig. 26. A G B H C K D L E F N
Fig. 27. A B G C D E H F
Fig. 28. A G B H D C E F
Fig. 29. A B E C D
Fig. 30. G C B A F E D
Fig. 31. A B C D E F
Fig. 32. A D B E C F G H
Fig. 33. A E B C F D
Fig. 33. D A E C G B
Fig. 34. H C D G B A F E
Fig. 35. F A E I D C B G H
Fig. 36. A G F E D C B
A G I C L B
Fig. 37. D H K E M F
Fig. 38. E A C D B
Fig. 39. F D A E C B
Fig. 40. A C B
Fig. 41. D F E G
Fig. 42. A C D B
Fig. 43. A G B C
Fig. 44. D E F
Fig. 45. C F E D A G B
Fig. 46. C E H D I B G F A
Fig. 47. D E N V S R A M O P Q B F C Z G X T L H
Fig. 48. E D A B C
Fig. 49. H G D E F
Fig. 50. F A E B
A F E C G F B B
Fig. 52. D C A B H G E F
Fig. 51. A D B C E
Fig. 53. E F D C B H G A
Fig. 54. A C G B D F E
Fig. 55. A G D I E F B H C
Fig. 56. F G K H I
Fig. 57. A B E C D
Fig. 58. B F I C A G H E D
Fig. 59. O N M L H G F E D K I C B A
Fig. 60. A I G B E H F D C
Fig. 61. L X A B C Z F O
F P O N
Fig. 62. G D E C H B K A M L I

*Coroll.*

From this propoſition you may learn how to add or ſubtract, any like figures, by the ſame method that is uſed in adding and ſubtracting of ſquares, in *Schol.* 47. 1.

## PROP. XXXII. Plate IV. *Fig.* 4.

*If two triangles* ABC, DCE *having two ſides proportional to two* (AB : AC : : DC. DE) *be ſo compounded or ſet together at one angle* ACD, *that their homologous ſides are alſo parallel* (AB *to* DC, *and* AC *to* DE,) *then the remaining ſides of thoſe triangles* (BC, CE) *ſhall be found placed in one ſtrait-line.*

For the angle A (*a*) = ACD (*a*) = D, and AB : AC a 29. 1.
(*b*) : : DC : DE, (*c*) therefore the angle B = DCE. b *hyp.*
Therefore the angle B+A (*d*) = ACE; but the angle c 6. 6.
B+A+ACB (*e*) = 2 right, (*f*) therefore the angle ACE d 2. *ax.*
+ACB=2 right; (*g*) therefore BCE is a right-line. e 32. 1.
*Which was to be demonſtrated.* f 1. *ax.*
g 14. 1.

## PROP. XXXIII. *Fig.* 7, 8.

*In equal circles* DBCA, HFGP, *the angles* BDC, FHG, *have the ſame ratio with the peripheries* BC, FG, *on which they ſtand; whether the angles be ſet at the centers (as* BDC, FHG) *or at the circumferences,* A, E: *And ſo likewiſe have the Sectors* BDC, FHG.

Draw the right-lines BC, FG. Make CI = CB, and GL = FG = LP, and join DI, HL, HP.

The arch BC (*a*)=CI, (*a*) alſo the arches FG, GL, LP, a 28. 3.
are equal; (*b*) therefore the angle BDC=CDI, (*b*) and the b 27. 3.
angle FHG=GHL=LHP. Therefore the arch BI is
the ſame multiple of the arch BC, as the angle BDI is
of the angle BDC. And in like manner is the arch FP,
the ſame multiple of the arch FG, as the angle FHP is of
the angle FHG. But if the arch BI $\sqsubset$, =, $\sqsupset$ FP,
(*c*) then likewiſe is the angle BDI $\sqsubset$, =, $\sqsupset$ FHP. c 27. 3.
Therefore as the arch BC : FG : : (*d*) the angle BDC : FHG d 6. *def.* 5.
(*e*) : : $\frac{BDC}{2}$ : $\frac{FGH}{2}$ (*f*) : : A : E. *Which was to be dem.* e 15. 5. f 20. 3.

Moreover, the angle BMC (*g*) = CNI; (*h*) and there- g 27. 3. h 24. 3.
fore the ſegment BCM=CIN. (*k*) Alſo the triangle k 4. 1.
BDC=CDI; (*l*) wherefore the ſector BDCM=CDIN. l 2. *ax.*

After the ſame manner are the ſectors F H G, G H L, L H P equal one to the other. Therefore ſince accordingly as the arch BI ⊏, =, ⊐ F G P, ſo is likewiſe the ſector BDI ⊏, =, ⊐ F H P; (*m*) thence
m 6. *def.* 5. it will be as the ſector BDC: FHG: : the arch B C: FG, *Which was to be demonſtrated.*

### *Coroll.*

a 11. 5. 1. Hence, *As ſector* (*a*) *is to ſector, ſo is angle to angle.*

2. *The angle* BDC *in the center, is to four right-angles, as the arch* B C, *on which it ſtands, to the whole circumference.*

For as the angle BDC is to a right-angle, ſo is the arch BC to a quadrant. Therefore BDC is to four right-angles, as the arch BC is to four quadrants, that is, to the whole circumference. Alſo, as the angle A : 2 right : : the arch BC : periphery.

3. Hence, *the arches* IL, BC, (Fig. 9.) *of unequal circles which ſubtend equal angles, whether at the centers, as* IAL *and* BAC, *or at the periphery, are like.*

For I L : periph. : : angle I A L (B A C) : 4 right. Alſo, Arch BC : periph. : : angle BAC : 4 right. Therefore IL : periph. : : BC : periph. And conſequently the arches IL and BC are ſimilar. Whence

4. *Two ſemidiameters* AB, AC, *cut off like arches* I L, BC *from concentric peripheries.*

*The End of the ſixth* BOOK.

# The SEVENTH BOOK OF EUCLIDE's ELEMENTS.

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## *Definitions.*

I. UNity is that, by which every thing that is, is called One.

II. Number is a multitude compoſed of units.

III. Part is a number of a number; the leſſer of the greater, when the leſſer meaſureth the greater.

*Every part is denominated from the number, by which it meaſures the number whereof it is a part; as* 4 *is called the third part of* 12, *becauſe it meaſures* 12 *by* 3.

IV. But when the leſſer number does not meaſure the greater, then the leſſer is call'd, not a part, but parts of the greater.

*All parts whatſoever are denominated from theſe two numbers, by which the greateſt common meaſure of the two numbers meaſures each of them; as* 10 *is ſaid to be two thirds of the number* 15; *becauſe the greateſt common meaſure, which is* 5, *meaſures* 10 *by* 2, *and* 15 *by* 3.

V, A multiple is a greater number compared with a leſſer, when the leſſer meaſures the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or that which differeth from an even number by unity.

VIII. A number evenly even, is that which an even number meaſureth by an even number.

IX. But a number evenly odd, is that which an even number meaſureth by an odd number.

X. A number only oddly odd, is that which an odd number meaſureth by an odd number.

XI. A prime (or firſt) number is that which is meaſured only by unity.

XII. Numbers prime the one to the other, are ſuch as only unity doth meaſure, being their common meaſure.

XIII. A compoſed number is that which ſome certain number meaſureth.

XIV. Numbers compoſed the one to the other, are thoſe, which ſome number, being a common meaſure to them both, doth meaſure.

*In this, and the preceding definition, unity is not a number.*

XV. One number is ſaid to multiply another when the number multiplied is ſo often added to it ſelf, as there are units in the number multiplying, and another number is produced,

*Hence in every multiplication unity is to the multiplier, as the multiplicand is to to the product.*

Obſ. *That many times, when any numbers are to be multiplied (as* A *into* B) *the conjunction of the letters denotes the product: So* $AB = A \times B$, and $CDE = C \times D \times E$.

XVI. When two numbers multiplying themſelves produce another, the number produced is called a plane number; and the numbers which multipled one another, are called the ſides of it: *So* 2 (C) × 3 (D) = 6 = CD *is a plane number.*

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a ſolid number; and the numbers multiplying one another, are called the ſides thereof: So 2 (C) × 3 (D) × 5 (E) = 30 = CDE *is a ſolid number.*

XVIII. A ſquare number is that which is equally equal; or, which is contained under two equal numbers. *Let* A *be the ſide of a ſquare; the ſquare is thus noted,* AA, *or* Aq.

XIX. A Cube is that number which is equally equal equally; or which is contained under three equal numbers. *Let* A *be the ſide of a Cube; the Cube is thus noted,* AAA, *or* Ac.

*In this definition, and the three foregoing, unity is number.*

XX. Numbers are proportional, when the firſt is the ſame multiple of the ſecond, as the third is of the fourth; or, the ſame part; or, when a part of the firſt number meaſures the ſecond, and the ſame part of the third meaſures the fourth, equally: and *vice verſa. So* A : B : : C : D. *that is,* 3 : 9 : : 5 : 15.

XXI. Like plane, and ſolid numbers, are thoſe which have their ſides proportional: *Namely, not all the ſides, but ſome.*

XXII. A perfect Number is that which is equal to its own parts.

*As* 6, *and* 28. *But a number that is leſs than it's parts is called an* Abounding *number, and one which is greater, a* Diminutive: *ſo* 12 *is an abounding,* 15 *a diminutive number.*

XXIII. One number is ſaid to meaſure another, by a third number, which when it either multiplies, or is multiplied by the meaſuring number, produces the number meaſured.

*In Diviſion, unity is to the quotient, as the diviſor is to the dividend.* Note, *that a number placed under another with a line between them, ſignifies diviſion:* So $\frac{A}{B} =$ A divided by B, and $\frac{CA}{B} =$ CXA divided by B.

Thoſe two numbers are called the Terms or Roots of a Proportion, than which leſſer cannot be found in the ſame proportion.

### *Poſtulates, or Petitions.*

1. THat numbers equal or multiple to any number may be taken at pleaſure.
2. That a number greater than any other whatſoever may be taken.
3. That Addition, Subtraction, Multiplication, Diviſion, and the Extractions of Roots or ſides of ſquare and cube numbers, be alſo granted as poſſible.

### *Axioms.*

1. WHatſoever agrees with one of many equal numbers, agrees likewiſe with the reſt.
2. Thoſe parts that are the ſame to the ſame part, or parts, are the ſame among themſelves.
3. Numbers that are the ſame parts of equal numbers, or of the ſame number, are equal among themſelves.

4. Thoſe numbers of which the ſame number or equal numbers, are the ſame parts, are equal amongſt themſelves.

5. Unity meaſures every number by the units that are in it, that is, by the ſame number.

6. Every number meaſures it ſelf by unity.

7. If one number multiplying another, produces a third, the multiplier ſhall meaſure the product by the multiplied; and the multiplied ſhall meaſure the ſame by the multipler.

*Hence, No prime number is either a plane, ſolid, ſquare, or cube number.*

8. If one number meaſures another, that number by which it meaſureth ſhall meaſure the ſame by the units that are in the number meaſuring, that is, by the number it ſelf that meaſures.

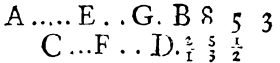
9. If a number meaſuring another, multiply that by which it meaſureth, or be multiplied by it, it produceth the number which it meaſureth.

10 How many numbers ſoever any number meaſureth, it likewiſe meaſureth the numbers compoſed of them.

11. If a number meaſures any number, it alſo meaſureth every number which the ſaid number meaſureth.

12. A number that meaſures the whole and a part taken away, doth alſo meaſure the reſidue.

## PROP. I.

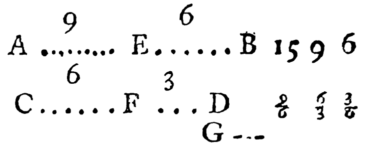


*Two unequal numbers* AB, CD, *being given, if the leſſer* CD, *be continually taken from the greater* AB (*and the reſidue* EB *from* CD, &c.) *by an alternate ſubtraction, and the number remaining never meaſures the precedent, till unity* GB *be taken; then are the numbers which were given* AB, CD, *prime the one to the other.*

If you deny it, let AB, CD, have a common meaſure, namely the number H; therefore H meaſuring CD, doth
a 11. *ax.* 7. (*a*) alſo meaſure AE; and (*b*) conſequently the remainder EB; (*a*) therefore it likewiſe meaſures CF, and (*b*) ſo the
b 12. *ax.* 7. remainder FD; (*a*) therefore it alſo meaſures EG. But it meaſured the whole EB, and (*b*) therefore it muſt meaſure that which remaineth GB, that is, a number mea-
c 9. *ax.* 1. ſures unity. (*c*) *Which is abſurd.*

## PROP. II.

*Two numbers* AB, CD *being given, not prime the one to the other, to find out their greateſt common meaſure* FD.



Take the leſſer number CD from the greater AB as often as you can. If nothing remains, (*a*) it is manifeſt that CD is the greateſt common meaſure. But if there remains ſomething (as EB) then take it out of CD, and the reſidue FD out of EB, and ſo forward till ſome number (FD) meaſure the ſaid EB, (*b*) for this will be, before you come to unity; FD ſhall be the greateſt common meaſure. a 6. *ax.* 7. b 1. 7.

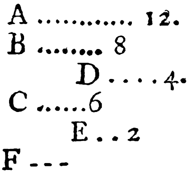
For FD (*c*) meaſures EB, and (*d*) therefore alſo CF; and (*e*) conſequently the whole CD; (*d*) therefore likewiſe AE; and ſo meaſures the whole AB. Wherefore it is evident that FD is a common meaſure. If you deny it to be the greateſt, let there be a greater (G); then whereas G meaſureth CD, it (*d*) muſt likewiſe meaſure AE, (*e*) and the reſidue EB, (*d*) as alſo CF, (*e*) and by conſequence the reſidue FD, (*g*) the greaterthe leſs. (*h*) *Which is abſurd.* c *conſtr.* d 11. *ax.* 7. e 12. *ax.* 7. d 11. *ax.* 7. e 12. *ax.* 7. g *ſuppoſ.* h 9. *ax.* 1.

### *Coroll.*

Hence, a number that meaſures two numbers, does alſo meaſure their greateſt common meaſure.

## PROP. III.

*Three numbers being given,* A, B, C, *not prime one to the other, to find out their greateſt common meaſure* E.



Find out D, the greateſt common meaſure of the two numbers A, B. If D meaſures C the third, it is clear that D is the greateſt common meaſure of all the three numbers. If D does not meaſure C, at leaſt D and C will be compoſed the one to the other, by the Coroll. of the Propoſition preceding. Therefore let E be the greateſt common meaſure of the ſaid numbers D and C, and it ſhall be the number which was required.

a *conſt.* For E (*a*) meaſures C and D, and D meaſures A and B;
b 11. *ax.* 7. therefore (*b*) E meaſures each of the numbers A, B, C:
neither ſhall any greater (F) meaſure them; for if you
c *cor.* 1 7. affirm that, (*c*) then F meaſuring A and B, does likewiſe
meaſure D their greateſt common meaſure; and in like
manner, F meaſuring D and C, does alſo meaſure E (*c*)
d *ſuppoſ.* their greateſt common meaſure, (*d*) the greater the leſs.
e 9. *ax.* 1. (*e*) *Which is abſurd.*

*Coroll.*

Hence, a number that meaſures three numbers, does alſo meaſure their greateſt common meaſure.

## PROP. IV.

A...... 6
B ........ 7
B .................. 18
a 4. *def.* 7. B .......... 9

*Every leſs number* A *is of every greater* B *either a part or parts.*

If A and B be prime to one another, (*a*) A ſhall be as many parts of the number B, as there are units
b 3. *def.* 7. in A (as $6=\frac{6}{7}$ of 7). But if A meaſures B, it is (*b*) plain
that A is a part of B (as $6=\frac{1}{3}$ of 18.) Laſtly, if A and B
c 4. *def.* 7. be otherwiſe compoſed to one another, (*c*) the greateſt
common meaſure ſhall determine how many parts A
does contain of B; as $6=\frac{2}{3}$ of 9.

## PROP. V.

A...... 6 D .... 4
6 6 4 4
B...... G ...... C 12 E.... H .... F 8.

*If a number* A *be a part of a number* BC, *and another number* D *the ſame part of another number* EF; *then both the numbers together* (A+D) *ſhall be the ſame part of both the numbers together* (BC+EF,) *which one number* A *is of one number* BC.

For if BC be reſolved into its parts BG, GC, equal
to A; and EF alſo into its parts EH, HF, equal to D;
a *hyp.* (*a*) the number of parts in BC ſhall be equal to the num-
b *conſt.* & ber of parts in EF. Therefore ſince A+D (*b*) = BG+
a *ax.* 1. EH = GC+HF, thence A+D ſhall be as often in BC+
EF, as A is in BC. *Which was to be demonſtrated.*

Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$, then (*c*) $2a = x$, and $2b = y$, therefore $2a + 2b = x + y$, therefore $a + b = \frac{x + y}{2}$ *Which was to be demonſtrated.* c 2. *ax.* 1.

## PROP. VI.

3 3 4 4
A...G...B 6 D....H.... E 8
C......... 9 F............ 12

*If a number* AB *be parts of a number* C, *and another number* DE *the ſame parts of another number* F; *then both numbers together* AB+DE *ſhall be of both numbers together* C+F *the ſame parts, that one number* AB *is of one number* C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The number of parts in both AB, DE, is equal by ſuppoſition; ſince then AG (*a*) is the ſame part of the number C, that DH is of the number F, AG + DH (*b*) ſhall be the ſame part of the compounded number C+F, that one number AG is of one number C. (*b*) In like manner GB+HE is the ſame part of the ſaid C+F, that one number GB is of one number C. (*c*) Therefore AB + DE is the ſame parts of C+F, that AB is of C. *Which was to be demonſtrated.* a *hyp.* b 5. 7. c 2. *ax.* 7.

Or thus. Let $a = \frac{2}{3}x$, and $b = \frac{2}{3}y$, and $x + y = g$, then, becauſe $3a = 2x$, and $3b = 2y$, is $3a + 3b = 2x + 2y = 2g$, therefore $a + b = \frac{2}{3}g = \frac{2}{3}x + y$.

## PROP. VII.

5 3
A..... E . . . B. 8
6 10 6
G......C..........F . . . . . . D 16

*If a number* AB *be the ſame part of a number* CD, *that a part taken away* AE *is of a part taken away* CF; *then ſhall the reſidue* EB *be the ſame part of the reſidue* FD *that the whole* AB *is of the whole* CD.

(*a*) Let EB be the ſame part of the number GC that AB is of CD, or AE of CF, (*b*) therefore AE+EB is the ſame part of CF+GC that AE is of CF, or AB of CD; (*c*) therefore GF=CD. Take away CF common to both, and (*d*) there remains GC=FD. (*e*) Wherefore a 1 *poſt.* 7. b 5. 7. c 6. *ax.* 1. d 3. *ax.* 1. e 2. *ax.* 7.

fore EB is the ſame part of the reſidue FD (GC) that the whole AB is of the whole CD. *Which was to be dem.*

Or thus. Let $a + b = x$; and $c + d = y$; and $x = 3y$,
f 1. 2. in like manner as $a = 3c$; I ſay $b = 3d$. *(f)* For $3c +$
g *hyp.* $3d = 3y = x$ *(g)* $= a + b$, take away from both $3c$ *(g)* $=$
$a$, and there remains $3d = b$. *Which was to be dem.*

## PROP. VIII.

6 2 4 2 2
A......H..G....E..L..B 16
18 6
C .................. F......D 24

*If in a number* AB *there be the ſame parts of a number* CD, *that a part taken away* AE, *is of a part taken away* CF; *the reſidue alſo* EB *ſhall be the ſame parts of the reſidue* FD, *that the whole* AB *is of the whole* CD.

Divide AB into AG, GB, parts of the number CD;
alſo AE into AH, HE, parts of the number CF; and
a 3. *ax.* take GL=AH=HE, (*a*) wherefore HG=EL. And
b *conſtr.* becauſe (*b*) AG=GB, (*c*) therefore HG=LB. Now
c 3. *ax.* 1. whereas the whole AG is the ſame part of the whole
CD that the part taken away AH is of the part taken
d 7. 7. away CF, (*d*) the reſidue HG or EL ſhall be the ſame
part alſo of the reſidue FD that AG is of CD. In like manner, becauſe GB is the ſame part of the whole CD, that HE or GL are of CF, (*d*) therefore the reſidue LB ſhall be the ſame part of the reſidue FD that GB is of the whole CD. Therefore EL+LB (EB) is the ſame parts of the reſidue FD, that the whole AB is of the whole CD. *Which was to be demonſtrated.*

Or thus more eaſily. Let $a + b = x$, and $c + d = y$.
e 9. *ax.* 7. Alſo $y = \frac{2}{3}x$ as well as $c = \frac{2}{3}a$; or, (*e*) which is the ſame
f 1. 2. $3y = 2x$; and $3c = 2a$. I ſay $d = \frac{2}{3}b$. For $3c +$
g 1. *ax.* 1. $3d$ *(f)* $= 3y = 2x$ *(f)* $= 2a + 2b$. (*g*) Therefore $3c + 3d$
h *hyp.* $= 2a + 2b$; take away from each $3c$ (*h*) $= 2a$, and (*k*)
k 3. *ax.* 1. there remains $3d = 2b$; (*l*) therefore $d = \frac{2}{3}b$. *Which*
l 8. *ax.* 7. *was to be demonſtrated.*

## PROP. IX.

A....4
4 4
B....G....C 8
D.....5
5 5
E.....H.....F 10

*If a number* A *be a part of a number* BC, *and another number* D *the ſame part of another number* EF; *then alternately what part or parts the firſt* A *is of the third* D, *the ſame part or parts ſhall the ſecond* BC *be of the fourth* EF.

A is ſuppoſed ⊐ D, therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts is put equal in both. But it is clear that BG is (*a*) the ſame part or parts of EH, that GC is of HF; (*b*) wherefore BC (BG+GC) is the ſame part or parts of EF (EH+HF) that BG alone (A) is of EH alone (D.) *Which was to be demonſtrated.*

a 1. *ax.* 7. & 4. 7.
b 5 *or* 6. 7.

Or thus. Let $a = \frac{b}{3}$, and $c = \frac{d}{3}$; or $3a = b$, and $3c = d$, then $\frac{c^*}{a} = \left( \frac{3c}{3a} = \right) \frac{d}{b}$.

* 15. 5.

## PROP. X.

*If a number* AB *be parts of a number* C, *and another number* DE *the ſame parts of another number* F, *then alternately, what parts or part the firſt* AB *is of the third* DE, *the ſame parts or part ſhall the ſecond* C *be of the fourth* F.

A . . G . . B 4
C . . . . . . 6
5 5
D. . . . . H . . . . . E 10
F . . . . . . . . . . . . . . . 15

AB is taken ⊐ DE, and C ⊐ F. Let AG, GB, and DH, HE, be parts of the numbers C and F, *viz.* as many in AB as in DE. It is manifeſt that AG is the ſame part of C, that DH is of F; (*a*) whence alternately AG is of DH, and likewiſe GB of HE, and (*b*) ſo conjointly AB of DE the ſame part, or parts, that C is of F. *Which was to be demonſtrated.*

a 9. 7.
b 5 & 9. 7.

Or thus. Let $a = \frac{2b}{3}$, and $c = \frac{2d}{3}$; or $3a = 2b$, and $3c = 2d$. Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

## PROP. XI.

*If a part taken away* AE *be to a part taken away* CF, *as the whole* AB *is to the whole* CD, *the reſidue alſo* EB *ſhall be to the reſidue* F D, *as the whole* AB *is to the whole* CD.

4 3
A . . . . E . . . B 7
8 6
C . . . . . . . . F . . . . . . D 14.

a 4. 7. Firſt, let AB be ⊐ CD; (*a*) then AB is either a part
b 20. *def* 7. or parts of the number CD; and likewiſe AE is (*b*) the
c 7. *or* 8. 7. ſame part or parts of CF; (*c*) therefore the reſidue EB
is the ſame part or parts of the reſidue FD that the whole AB is of the whole CD, (*b*) and ſo AB : CD : : EB : FB. But if AB be ⊏ CD, then according to what is already ſhewn, will CD : AB : : FD : EB, therefore by inverſion AB : CD : : EB : FD. *W.W.D.*

## PROP. XII.

A, 4. C, 2. E, 3.
B, 8. D, 4. F, 6.

*If there be numbers, how many ſoever, proportional* (A : B : : C : D : : E : F;) *then as one of the antecedents* A *is to one of the conſequents* B, *ſo ſhall all the antecedents* (A+C+E) *be to all the conſequents* (B+D+F.)

Firſt, let A, C, E, be ⊐ B, D, F; then, becauſe
a 20. *def.* 7. of the ſame proportions, (*a*) ſhall A be the ſame part or
b 5. & 6, 7. parts of B that C is of D; (*b*) and likewiſe conjointly A+
C ſhall be the ſame part or parts of B+D, that A alone
is of B alone. In the like manner A+C+E is the ſame
c 20. *def.* 7. part or parts of B+D+F that A is of B. (*c*) Therefore
A+C+E : B+D+F : : A : B. But if A, C, E, be put greater than B, D, F, the ſame thing may be ſhewn by inverſion. *W.W.D.*

## PROP. XIII.

A, 3. C, 4.
B, 9. D, 12.

*If there be four numbers proportional* (A : B : : C : D) *then alternately they ſhall alſo be proportional,* (A : C : : B : D.)

Firſt, let A and C be ⊐ B and D, and A ⊐ C. By
a 20. *def.* 7. reaſon of the ſame proportion (*a*) A ſhall be the ſame
b 9. & 10. part or parts of B, that C is of D. (*b*) Therefore alternately A is the ſame part or parts of C, that B is of D; and ſo A : C : : B : D. But if A be ⊏ C, and A and C ſuppoſed ⊏ B and D, it will come to the ſame thing by inverting the proportions. *W.W.D.*

## PROP. XIV.

A, 9, D, 6.
B, 6. E, 4.
C, 3, F, 2.

*If there be numbers, how many ſoever,* A, B, C, *and as many more equal to them in number, which may be compared two and two in the ſame proportion* (A : B : :

: : D : E *and* B : C : : E : F ;) *they ſhall alſo by equality, be in the ſame proportion* (A : C : : D : F.)

For becauſe A : B : : D : E, (*a*) therefore alternately is a 13. 7.
A : D : : B : E : : (*a*) C : F ; (*a*) therefore again, by permutation, A : C : : D : F. *Which was to be demonſtrated.*

## PROP. XV.

*If an unite meaſure any number B, and another number D equally meaſure ſome other number E ; alternately alſo ſhall an unite meaſure the third number* D, *as often as the ſecond* B *doth the fourth* E.

1. D..2
B...3.E......6.

For ſeeing 1 is the ſame part of B, that D is of E ;
(*a*) therefore alternately ſhall 1 be the ſame part of D, a 9. 7.
that B is of E. *Which was to be demonſtrated.*

## PROP. XVI.

*If two numbers* A, B, *mutually multiplying themſelves, produce any numbers* AB, BA ; *the numbers produced* AB, *and* BA, *ſhall be equal the one to the other.*

B, 4. A, 3.
A, 3. B, 4.
AB, 12. BA, 12.

For becauſe AB = A × B, (*a*) therefore ſhall 1 be as a 15. *def.* 7.
often in A, as B in A B, (*b*) and by conſequence alter- b 15. 7.
nately 1 ſhall be as often in B as A in A B. But becauſe
BA = B × A, (*a*) therefore ſhall 1 be as often in B, as
A in BA, therefore as often as 1 is in AB, ſo often is 1
in BA, and (*c*) ſo AB = BA. *Which was to be demon-* c 4. *ax.* 7.
*ſtrated.*

## PROP. XVII.

*If a number* A *multiplying two numbers* B, C, *produce other numbers* AB, AC ; *the numbers produced of them ſhall be in the ſame proportion that the numbers multiplied are.* (AB : AC : : B : C.)

A, 3.
B, 2. C, 4.
AB, 6. AC, 12.

For ſince AB = A × B (*a*) therefore ſhall 1 be as often a 15. *def.* 7
in A as B in AB. (*a*) Likewiſe becauſe AC = A × C
ſhall 1 be as often in A, as C in AC, and ſo alſo
B as often in AB as C in AC ; (*b*) wherefore B : AB : : b 20. *def.* 7.
C : AC, (*c*) and therefore alſo alternately B : C : : AB : c 13. 7.
AC. *Which was to be demonſtrated.*

ROP.

## PROP. XVIII.

C, 5. C, 5.
A, 3. B, 9.
AC, 15. BC, 45.

*If two numbers* AB, *multiplying any number* C, *produce other numbers* AC, BC; *the numbers produced of them ſhall be in the ſame proportion that the numbers multiplied are.* (A : B : : AC : BC.)

a 16. 7. For (*a*) AC=CA, and BC (*a*) = CB ; ſo the ſame C
b 17. 7. multiplying A and B produceth AC and BC ; (*b*) therefore A : B : : AC : BC. *Which was to be demonſtrated.*

*Schol.*

Hence is deduced the vulgar manner of reducing fractions ($\frac{3}{5}$, $\frac{7}{9}$,) to the ſame denomination. For multiply 9 both by 3 and 5, and they produce $\frac{27}{45} = \frac{3}{5}$; becauſe by this 3 : 5 : : 27 : 45. Likewiſe multiply 5 by 7 and 9, there ariſes $\frac{35}{45} = \frac{7}{9}$; becauſe 7 : 9 : : 35 : 45.

## PROP. XIX.

A, 4. B, 6. C, 8. D, 12.
AD, 48. BC, 48.

*If there be four numbers in proportion* (A : B : : C : D.) *the rectangle or number produced of the firſt and fourth* (AD) *is equal to the rectangle or number which is produced of the ſecond and third* (BC.) *And if the number which is produced of the firſt and fourth* (AD) *be equal to that produced of the ſecond and third* (BC) *thoſe four numbers ſhall be in proportion* (A : B : : C : D.)

a 17. 7. 1. *Hyp.* For AC : AD (*a*) : : C : D (*b*) : : A : B (*c*)
b *hyp.* : : AC : BC ; (*d*) therefore AD = BC. *Which was to*
c 18. 7. *be demonſtrated.*

d 9. 5. 2. *Hyp.* Becauſe (*e*) AD = BC, therefore AC : AD (*f*)
e *hyp.* : : AC : BC. But AC : AD (*g*) : : C : D, and AC : BC
f 17. 5. (*h*) : : A : B ; (*k*) therefore C : D : : A : B. *Which*
g 17. 7. *was to be demonſtrated.*
h 18. 7.
k 11. 5.

## PROP. XX.

*If there are three numbers in proportion* (A : B : : B : C) *the number contained under the extremes* (A C) *is equal to the ſquare made of the middle* (BB). *And if the number contained under the extremes be eqnal to that* (Bq,) *produced of the middle, thoſe three numbers ſhall be in proportion* A : B : : B : C.

A. B. C.
4. 6. 9.
AC, 36. BB, 36.
D, 6.

1. *Hyp.* For take D=B, (*a*) therefore A : B : : D (B) : C (*b*) wherefore AC = BD, (*a*) or BB. *Which was to be demonſtrated.*

2. *Hyp.* Becauſe AC (*c*) = BD, (*b*) therefore A : B : : D (B) : C. *Which was to be demonſtrated.*

a 1. *ax.* 7.
b 19. 7.
c *hyp.*
d 19. 7.

## PROP. XXI.

*Numbers* A B, C D, *being the leaſt of all that have the ſame proportion with them* (E, F,) *equally meaſure the numbers* E, F, *having the ſame proportion with them; the greater* AB *the greater* E, *and the leſſer* CD, *the leſſer* F.

A...G..B 5. E..........10.
C..H.D 3. F......6.

For AB : CD (*a*) : : E : F ;(*b*) therefore alternately AB : E : : CD : F ; (*c*) therefore AB is the ſame part or parts of E that CD is of F ; but parts it cannot be, for if ſo, then let AG, GB, be parts of the number E ; and CH, HD, parts of the number F, (*c*) therefore AG : E : : CH : F, and by inverſion AG : CH (*d*) : : E : F, (*e*) : : AB : CD ; therefore AB, CD, are not the leaſt in their proportion ; which is contrary to the hypotheſis. Therefore, &c:

a *hyp.*
b 13. 7.
c 20. *def.*
d 13. 7.
e *hyp.*

## PROP. XXII.

*If there are three numbers* A, B, C; *and other numbers equal to them in number,* D, E, F; *which may be compared two and two in the ſame proportion: and if alſo the proportion of them be perturbed* (A : B : : E : F, *and* B : C : : D : E.) *then by equality they ſhall be in the ſame proportion* (A : C : : D : F.)

A, 4. D, 12.
B, 3. E, 8.
C, 2. F, 6.

For

a *hyp.* For becauſe A : B (*a*) : : E : F, therefore ſhall AF =
b 19. 7. BE ; and becauſe B : C : : (*a*) D : E, (*b*) therefore BE=
c 1 *ax.* 1. CD, (*c*) and conſequently AF = CD. (*d* Therefore
d 19. 7. A : C : : D : F. *Which was to be demonſtrated.*

## PROP. XXIII.

A, 9. B, 4.
C ---- D ---
E- - -

*Numbers prime the one to the other, A, B, are the leaſt of all numbers that have the ſame proportion with them.*

If it be poſſible, let C and D be leſs than A and B,
a 21. 7. and in the ſame proportion ; (*a*) therefore C meaſures A
equally as D meaſures B, ſuppoſe by the ſame number E ;
b 23. *def.* 7. and ſo C ſhall be (*b*) as often in A as 1 is in E, (*c*) like-
c 15. 7. wiſe alternately, E as often in A as 1 in C. By the
like reaſoning as many times as 1 is in D, ſo many times
ſhall E be in B. Therefore E meaſures both A and B ;
which conſequently are not prime the one to the other,
*contrary to the hypotheſis.*

## PROP. XXIV.

A, 9. B, 4.
C - - -
D - - - E - -

*Numbers A, B, being the leaſt of all that have the ſame proportion with them, are prime the one to the other.*

If it be poſſible, let A and B have a
common meaſure C ; and let the ſame meaſure A by D,
a 9. *ax.* 7. and B by E ; (*a*) therefore CD=A, (*b*) and CE=B. (*b*)
b 17. 7. Wherefore A : B : : D : E. But D and E are leſſer than
A and B, as being but parts of them. Therefore A
and B are not the leaſt in their proportion, *againſt the Hypotheſis.*

## PROP. XXV.

A, 9. B, 4.
C, 3. D- -

*If two numbers A, B, are prime the one to the other, the number C meaſuring one of them A, ſhall be prime to the number B.*

For if you affirm any other D to meaſure the numbers
a 11. *ax.* 7. B and C, (*a*) then D meaſuring C does alſo meaſure A,
and conſequently A and B are not prime the one to the
other. *Which is againſt the Hypotheſis.*

PROP.

## PROP. XXVI.

*If two numbers* A, B, *are prime to any number* C, *the number alſo produced of them* AB, *ſhall be prime to the ſame* C.

A, 5. C, 8.
B, 3.
AB, 15. E ----
F ----

If it be poſſible, let the number E be a common meaſure
to A B, and C; and let $\frac{AB}{E}$ be = F; (*a*) thence AB a 9. *ax.* 7.
= EF; (*b*) wherefore alſo F : A : : B : F, But becauſe A b 19. 7.
is prime to C, which E meaſures, (*c*) therefore E and A c 25. 7.
are prime to one another, (*d*) and ſo leaſt in their own d 23. 7.
proportion, (*e*) and conſequently they muſt meaſure B and e 21. 7.
F; namely F ſhall meaſure B, and A ſhall meaſure F.
Therefore ſeeing E meaſures both B and C, they ſhall
not be prime to one another: *Contrary to the Hypotheſis.*

## PROP. XXVII.

*If two numbers* A, B, *are prime to one another, that alſo which is produced of one of them* (Aq) *ſhall be prime to the other* B.

A, 4. B, 5.
Aq, 16.
D, 4.

Take D = A; therefore both D, and A are prime to a 1. *ax.* 7.
B; (*b*) therefore AD or Aq is prime to B. *Which was* b 26. 7.
*to be demonſtrated.*

## PROP. XXVIII.

*If two numbers* A, B, *are prime to two numbers* C, D, *each to either of both, the numbers alſo produced by multiplying them* AB, CD, *ſhall be prime to one another.*

A, 5. C, 4.
B, 3. D, 2.
AB, 15. CD, 8.

For becauſe A and B are prime to C, (*a*) therefore
ſhall AB alſo be prime to the ſame. And for the ſame a 26. 7.
reaſon ſhall AB be prime to D. (*b*) Therefore AB is
prime to CD. *Which was to be demonſtrated.* b 26. 7.

## PROP. XXIX.

*If two numbers* A, B, *are prime to one another, and each multiplying itſelf produces another number* (Aq, *and* Bq;) *then the numbers pro-*

A, 3. B, 2.
Aq, 9. Bq, 4.
Ac, 27. Bc, 8.

*duced of them* (Aq, Bq) *ſhall be prime to one another. And if the numbers given at firſt* A, B, *multiplying the ſaid produced numbers* (Aq, Bq,) *produce others* (Ac, Bc,) *thoſe numbers alſo ſhall be prime to one another: And ſo on.*

For becauſe A is prime to B, (*a*) therefore Aq ſhall
a 27. 7. be prime to B, and Aq being prime to B, (*a*) therefore
Aq ſhall be alſo prime to Bq. Again, becauſe A is as well
b 28. 7. prime to B and Bq, as Aq is to the ſaid B and Bq, (*b*)
therefore ſhall A×Aq, that is, Ac, be prime to B×Bq,
that is, to Bc: And ſo forth of the reſt. *W.W.D.*

## PROP. XXX.

8 5
A ........ B . . . . . C 13 . D - - - -

*If two numbers* AB, BC, *be prime the one to the other; then both added together* (AC) *ſhall be prime to either of them* AB, BC. *And if both added together* AC *be prime to any one of them* AB, *the numbers alſo given in the beginning* AB, BC, *ſhall be prime to one another.*

1. *Hyp.* For if you would have AC, AB to be com-
a 12. *ax.* 7. poſed, let D be the common meaſure: (*a*) this ſhall mea-
ſure the reſidue BC: And therefore AB, BC, are not
prime to one another; *which is againſt the Hypotheſis.*

2. *Hyp.* AC, AB, being taken prime to one another,
b 10. *ax.* 7. let D be the common meaſure of AB, BC. (*b*) But ſeeing
that meaſures the whole AC, therefore AC, AB, are not
prime to one another; *contrary to the Hypotheſis.*

### *Coroll.*

Hence, a number, which being compounded of two, is prime to one of them, is alſo prime to the other.

## PROP. XXXI.

A, 5. B, 8.

*Every prime number* A *is prime to every number* B, *which it meaſureth not.*

For if any common meaſure doth meaſure both, A,
a 11. *def.* 7. B, (*a*) then A will not be a prime number; *contrary to the*
*Hypotheſis.*

## PROP. XXXII.

A, 4. D, 3.
B, 6. E, 8.
AB, 24.

*If two numbers* A, B, *multiplying one another produce another* A B, *and ſome prime Number* D, *meaſure the number produced of them* AB; *then ſhall it alſo meaſure one of thoſe numbers,* A, *or* B, *which were given at the beginning.*

Suppoſe the number D not to meaſure the number A,
and let $\frac{AB}{D}$ be = E, (*a*) then AB = DE; (*b*) whence D : a 9. *ax.* 7.
A : : B : E. (*c*) But D is prime to A ; (*d*) therefore D and b 19. 7. c *hyp. and*
A are the leaſt in their proportion ; (*e*) and conſequently 31. 7.
D meaſures B as often as A meaſures E. *The propoſition* d 23. 7.
*therefore is evident.* e 21. 7.

## PROP. XXXIII.

*Every compoſed number* A, *is meaſured by* A, 12.
*ſome prime number* B. B, 2.

Let one or more numbers (*a*) meaſure A, of which let a 13. *def.* 7.
the leaſt be B; that ſhall be a prime number: For if it
be ſaid to be compoſed, then ſome (*a*) leſſer number
ſhall meaſure it, (*b*) which ſhall alſo conſequently meaſure b 11. *ax.* 7.
A. Wherefore B is not the leaſt of thoſe which meaſure
A, *contrary to the Hyp.*

## PROP. XXXIV.

*Every number* A, *is either a prime, or meaſured* A, 9.
*by ſome prime number.*

For A is neceſſarily either a prime or a compoſed
number. If be a prime, 'tis that we affirm. If com-
poſed, (*a*) then ſome prime number meaſureth it. *Which* a 33. 7.
*was to be demonſtrated*

## PROP. XXXV.

A, 6. B, 4. C, 8.
D, 2. H - - I - - K - - - .
E, 3. F, 2. G, 4, L - - -

*How many numbers ſoever* A, B, C, *being given to find the leaſt numbers* E, F, G, *that have the ſame proportion with them.*

If A, B, C, be prime to one another, (*a*) they ſhall a 23 7.
be the leaſt in their proportion. If they be compoſed,
(*b*) let their greateſt common meaſure be D, which let b 3. 7.
meaſure them by E, F, G. Theſe are then the leaſt
in the proportion A, B, C.

For D × E, F, G, (*c*) produceth A, B, C, (*d*) therefore c 9 *ax.* 7.
theſe and thoſe are in the ſame proportion But allow d 17. 7.
other numbers H, I, K, to be the leaſt in the ſame pro-
portion

e 21. 7. portion ; (*e*) which ſhall therefore equally meaſure A, B,
f 9. *ax.* 7. C, namely by the number L, (*f*) therefore L × H, I, K,
g 1. *ax.* 1. ſhall produce A, B, C, (*g*) and conſequently ED = A =
h 19. 7. HL ; (*h*) from whence E : H : : L : D. But E (*k*) $\sqsubset$
k *ſuppoſ.* H ; (*l*) therefore L $\sqsubset$ D, and ſo D is not the greateſt
l 20. *def.* 7. common meaſure of A, B, C. *Which is againſt the Hypotheſis.*

*Coroll.*

Hence, The greateſt common meaſure of how many numbers ſoever, meaſures them by the numbers which are leaſt of all that have the ſame proportion with them. Whence is derived the vulgar method of reducing fractions to their leaſt terms.

## PROP. XXXVI.

*Two numbers being given,* A, B, *to find out the leaſt number which they meaſure.*

A, 5. B, 4.
AB, 20.
D - - - - - -
E - - -F- - -

1. *Caſe.* If A and B be prime the one to the other, AB is the number required. For it is manifeſt that A and B meaſure AB. If it be poſſible, let A and B meaſure ſome other number D
a 9. *ax.* 7. & 1. *ax.* 1. $\sqsupset$ AB, ſuppoſe by E, and F ; (*a*) therefore AE = D =
b 19. 7. BF, (*b*) and ſo A : B : : F : E. But becauſe A and B (*c*)
c *hyp.* are prime one to the other, (*d*) and ſo leaſt in their pro-
d 23. 7. portion, A ſhall (*e*) equally meaſure F as B does E. But
e 21. 7. B : E (*f*) : : A B : A E (D.) (*g*) Therefore AB ſhall alſo
f 17. 7. meaſure D, which is leſs than it ſelf. *Which is abſurd.*
g 20. *def.* 7.

A, 6. B, 4, F - - - -
C, 3. D, 2. G- - -H- - -
h 35. 7. AD, 12.

2. *Caſe.* But if A and B be compoſed one to another, (*h*) let there be found C and D the leaſt
k 19. 7. in the ſame proportion. (*k*) Therefore AD = BC ; and AD or BC ſhall be the number ſought.
l 7. *ax.* 7. For it is (*l*) plain that B and A meaſure AD or BC. Conceive A and B to meaſure F $\sqsupset$ AD, namely A by
m 9. *ax.* 7. G, and B by H, (*m*) therefore AG = F = BH ; (*n*)
n 19. 7. whence A : B : : H : G (*o*) : : C : D, (*p*) and conſequent-
o *conſtr.* ly C equally meaſures H as D does G. But D : G (*q*)
p 21 7. : : A D : A G (F,) therefore AD (*r*) meaſures F, the
q 17. 7. greater the leſs. *Which is abſurd.*
r 20. *def.* 7.

*Coroll.*

Hence, If two numbers multiply the leaſt that are in the ſame proportion, the greater the leſs, and the leſs the greater, the leaſt number which they meaſure ſhall be produced.

## PROP. XXXVII.

*If two numbers* A, B, *meaſure any number* CD, *the leaſt number which they meaſure,* E, *ſhall alſo meaſure the ſame* CD.

A, 2. B, 3.
E......6.
C----F---D.

If you deny it, take E from CD as often as you can, and leave FD⊐E, therefore ſeeing A and B (*a*) meaſure E, (*b*) and E meaſures CF, (*c*) likewiſe A and B will meaſure CF. But (*a*) they meaſure the whole CD; (*d*) therefore alſo they meaſure the reſidue FD; and conſequently E is not the leaſt which A and B meaſure: *Contrary to the Hypotheſis.*

a *hyp.*
b *conſtr.*
c 11. *ax.* 7.
d 12. *ax.* 7.

## PROP. XXXVIII.

*Three numbers being given,* A, B, C, *to find out the leaſt which they meaſure.*

A, 3. B, 4. C, 6.
D, 12.

(*a*) Find D the leaſt that two of them A and B meaſure; which if the third C alſo meaſure, it is manifeſt that D is the number ſought. But if C doth not meaſure D, let E be the leaſt that C and D meaſure, E ſhall be the number required. a 36. 7.

For it appears by the 11. *ax.* 7. that A, B, C, meaſure E; and it is eaſily ſhewn that they meaſure no other F leſs than E.

A, 2. B, 3. C, 4.
D, 6. E, 12.
F---

For if you affirm they do, (*b*) then D meaſures F, (*b*) and conſequently E meaſures the ſame F, the greater the leſs. *Which is abſurd.* b 37. 7.

*Coroll.*

Hence it appears, that if three numbers meaſure any number, the leaſt alſo, which they meaſure, ſhall meaſure the ſame.

## PROP. XXXIX.

A, 12.
B, 4. C, 3.

*If any number* B, *meaſures a number* A, *the number meaſured* A, *ſhall have a part* C *denominated of the number meaſuring* B.

a *hyp.* For becauſe $\frac{A}{B}$ (*a*) = C, (*b*) therefore A = BC (*c*) and
b 9. *ax.* 7. c 7. *ax.* 7. $\frac{A}{C}$ = B. *Which was to be demonſtrated.*

## PROP. XL.

A, 15.
B, 3. C, 5.

*If a number* A, *have any part whatſoever* B, *the number* C, *from which the part* B *is denominated, ſhall meaſure the ſame.*

a *hyp.* & 9. *ax.* 7. For ſince BC (*a*) = A, (*b*) therefore $\frac{A}{C}$ = B. *Which*
b 7. *ax.* 7. *was to be demonſtrated.*

## PROP. XLI.

$\frac{1}{2}$ G, 12.
$\frac{1}{3}$ H - - -
$\frac{1}{4}$

*To find out a number* G, *the leaſt that can have given parts,* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

a 38. 7. (*a*) Let G be found the leaſt which the denominators
b 39. 7. 2, 3, 4, meaſure; (*b*) it is evident that G has the parts
c 40. 7. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. If it be poſſible let $H \sqsupset G$ have the ſame
parts; (*c*) therefore 2, 3, 4, meaſure H; and ſo G is not the leaſt which 2, 3, 4, meaſure: *againſt the ſuppoſition.*

### *The End of the ſeventh* BOOK.

# The EIGHTH BOOK OF EUCLIDE's ELEMENTS.

## PROP. I.

A, 8. B, 12, C, 18. D, 27.
E - F - - G - - - H - - - -

*IF there be divers numbers how many ſoever in continual proportion,* A, B, C, D, *and their extremes* A, D, *prime to one another; then thoſe numbers* A, B, C, D, *are the leaſt of all numbers that have the ſame proportion with them.*

For, if it be poſſible, let there be as many others E, F, G, H, leſs than A, B, C, D, and in the ſame proportion with them. (*a*) Therefore from equality A : D : : E : H, and ſo A and D, which are prime numbers, (*b*) and conſequently the leaſt in their proportion, (*c*) equally meaſure E and H, which are leſs than themſelves. *Which is abſurd.*

a 14. 7.
b 23. 7.
c 21. 7.

## PROP. II.

1
A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.
Ac, 8. AqB, 12. BqA, 18. Bc, 27.

*To find out the leaſt numbers continually proportional, as many as ſhall be required, in the proportion given of* A *to* B.

Let A and B be the leaſt in the proportion given; Then Aq, AB, Bq, ſhall be the three leaſt in the ſame continual proportion that A is to B.

For AA : AB (*a*) : : A : B (*b*) : : AB : BB. Likewiſe becauſe A and B are prime one to the other, (*c*) ſhall Aq, Bq, be alſo prime to one another, (*d*) and ſo Aq, AB, Bq, are ∺ the leaſt in the proportion of A to B. More-

a 17. 7.
b 24. 7.
c 29. 7.
d 1, 8.

Moreover, I ſay Ac, AqB, BqA, BC, are the four leaſt
e 17. 7. in the proportion of A to B. For AqA : AqB (*e*) : : A :
f 29. 7. B (*e*) : : ABA (AqB.) : ABB, (*e*) and A : B : : ABqA :
g 1. 8. BBB (Bc.) Therefore ſince Ac, and Bc, are (*f*) prime to
one another, likewiſe (*g*) ſhall Ac, AqB, BqA, Bc be the
four leaſt ∺ in the proportion of A to B. In the ſame
manner may you find out as many proportional numbers
as you pleaſe. *Which was to be done.*

### *Coroll.*

1. Hence, If three numbers, being the leaſt, are proportional, their extremes ſhall be ſquares ; if four, cubes.

2. The extremes of any number of proportionals found by this propoſition, if ſuch proportionals are the leaſt of all in a given ratio, are prime to one another.

3. Two numbers, being the leaſt in a given ratio, meaſure all the mean numbers of proportionals be they ever ſo many, provided they are the leaſt in the ſame proportion ; becauſe they ariſe from the multiplication of them into certain other numbers.

4. Hence it alſo appears by the conſtruction, that the ſeries of numbers 1, A, Aq, Ac. ; 1, B, Bq, Bc ; Ac, AqB, BqA, Bc, conſiſt of an equal multitude of numbers ; and conſequently, the extreme numbers of how many ſoever the leaſt continually proportionals are the laſt of as many other continually proportionals from unity ; thus the extremes Ac, Bc, of the continual proportionals Ac, AqB, BqA, Bc, are the laſt of as many proportionals from unity 1, A, Aq, Ac, and 1, B, Bq, Bc.

5. 1, A, Aq, Ac ; and B, BA, AqB ; and Bq, BqA are ∺ in the ratio of 1 to A. Alſo B, Bq, Bc ; and A, AB, BqA ; and Aq, AqB are ∺ in the ratio of 1 to B.

## PROP. III.

A, 8. B, 18. C, 12. D, 27. *If there be numbers continually proportional, how many ſoever,* A, B, C, D, *being alſo the leaſt of all that have the ſame proportion with them* ; *their extremes* A, D, *are prime to one another.*

a 2. 8. For if there be (*a*) found as many numbers the leaſt in
the proportion of A to B, they ſhall be no other than
A, B, C, D ; therefore, by the ſecond *Coroll.* of the precedent *prop.* the extremes A and D are prime to one another. *Which was to be demonſtrated.*

## PROP. IV.

A, 6. B, 5. C, 4, D, 3.
H, 4. F, 24. E, 20. G, 15
I - - K - - L - - -

*Proportions how many ſoever being given in the leaſt numbers (* A, *to* B, *and* C *to* D) *to find out the leaſt numbers continually proportional in the proportions given.*

(*a*) Find out E, the leaſt number which B and C meaſure; and let B meaſure E (*b*) as often as A does another F, *viz.* by the ſame number H. (*b*) Alſo let C meaſure the ſaid E as often as D meaſures another G, then F, E, G, ſhall be the leaſt in the proportions given. For AH (*c*) = F, and BH (*c*) = E; (*d*) therefore A : B : : AH : BH (*e*) : : F : E. In like manner C : D : : E : G; therefore F, E, G, are continually proportional in the proportions given. And they are moreover the leaſt in the ſaid proportions; for conceive other numbers I, K, L, to be the leaſt; (*f*) then A and B muſt equally meaſure I and K, (*f*) and C and D likewiſe K and L; and ſo B and C meaſure the ſame K. (*g*) Wherefore alſo E meaſures the ſame number K, which is leſs then it ſelf. *Which is abſurd.*

a 36. 7.
b 3 *poſt.* 7.
c 9. *ax.* 7.
d 18. 7.
e 7. 5.
f 21. 7.
g 37. 7.

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the leaſt continually in the proportions of A to B, and C to D. Then if E meaſures I, (*h*) take another number K, which may be equally meaſured by F; and thoſe four numbers H, G, I, K, ſhall be continually the leaſt in the given proportions; which we need go no other way to prove than we did in the firſt part.

h 3 *poſt.* 7.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 20. I, 15.
M, 48. L, 40. K, 30. N, 105.

If E doth not meaſure I, let K be the leaſt which E and I meaſure; and as often as I meaſures K, let G as often meaſure L, and H alſo M, ſo likewiſe let F meaſure N as often as E meaſures K. The four numbers M, L, K, N, ſhall be the leaſt continually in the given proportions; which we may demonſtrate as before.

## PROP. V.

C, 4. E, 3.
D, 6. F, 16. ED, 18.
CD, 24. EF, 48.

*Plane numbers* CD, EF, *are in that proportion to one another which is compoſed of their ſides.*

a 17. 7. For becauſe CD : DE (*a*) : : C : E ; (*a*) and ED : EF
b 20 *def.* 5.
c 11. 5. : : D : F and $\frac{CD}{EF}$ (*b*) $= \frac{CD}{ED} \times \frac{ED}{EF}$ (*c*) then ſhall be the
proportion $\frac{CD}{EF} = \frac{C}{E} \times \frac{D}{F}$. *Which was to be demon.*

## PROP. VI.

A, 16. B, 24. C, 36. D, 54. E, 81.
F, 4. G, 6. H, 9.

*If there be numbers continually proportional how many ſoever,* A, B, C, D, E, *and the firſt* A *does not meaſure the ſecond* B, *neither ſhall any of the other meaſure any one of the reſt.*

a 20. *def.* 7. Since A does not meaſure B, (*a*) neither ſhall any one
meaſure that which next follows; Becauſe A : B : : B : C
b 35. 7. : : C : D, &c. (*b*) Take three numbers, F, G, H, the leaſt
in the proportion of A to B, therefore ſince A does not
c 5. 7. meaſure B, (*a*) neither ſhall F meaſure G, (*c*) therefore
F is not unity. But F and H are prime one to another;
d 3. 3. therefore (*d*) ſince by equality A : C : : F : H, and F
does not meaſure H, (*a*) neither ſhall A meaſure C; and
conſequently neither ſhall B meaſure D, nor C meaſure
e 14. 7. E, &c. becauſe A : C (*e*) : : B : D (*e*) : : C : E, &c.
In like manner four or five numbers being taken the
leaſt in the proportion of A to B, it may be ſhewn that
A does not meaſure D and E; nor does B meaſure E and
F, &c. Wherefore none of them ſhall meaſure any other.
*Which was to be demonſtrated.*

## PROP. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

*If there be numbers continually proportional how many ſoever* A, B, C, D, E, *and the firſt* A *meaſures the laſt* E, *it ſhall alſo meaſure the ſecond* B.

a 6. 7. If you deny that A meaſures B, (*a*) then neither ſhall it meaſure E; *Which is contrary to the Hyp.*

PROP.

## PROP. VIII.

A, 24. C, 36. D, 54. B, 81.
G, 8. H, 12. I, 18. K, 27.
E, 32. L, 48. M, 72. F, 108.

*If between two numbers* A,B, *there fall mean numbers in continual proportion* C, D; *as many mean numbers in continual proportion as fall between them, ſo many mean numbers alſo* L, M, *in continual proportion, ſhall fall between two other numbers* E, F, *which have the ſame proportion with them* (L. M,)

(*a*) Take G, H, I, K, the leaſt ∺ in the proportion of A to C; (*b*) by equality it ſhall be G : K : : A : B (*c*) : : E : F. But G, and K (*d*) are prime one to another. (*e*) Wherefore G meaſures E as often as K does F. Let H meaſure L, and I likewiſe M by the ſame number; (*f*) therefore E, L, M, F, are in ſuch proportion as G, H, I, K, that is, as A, B, C, D. *Which was to be demonſtrated.*

a 35. 7.
b 14. 7.
c *hyp.*
d 3. 8.
e 21. 7.
f *conſtr.*

## PROP. IX.

1.
E, 2. F, 3.
G, 4. H, 6. I, 9.
A, 8. C, 12. D, 18. B, 27.

*If two numbers* A, B *are prime to one another, and mean numbers in continual proportion* C, D, *fall between them; as many mean numbers in continual proportion as fall between them, ſo many means alſo in continual proportion* (E, G; and F, I) *ſhall fall between either of them and unity.*

It is evident, that 1, E, G, A, and 1, F, I, B, are ∺, and as many as A, C. D, B, namely by the 4th Coroll. 2. 8. *Which was to be demonſtrated.*

## PROP. X.

A, 8. I, 12. K, 18. B, 27.
E, 4. DF, 6. G, 9.
D, 2. F, 3.
1.

*If between two numbers* A, B, *and an unit, numbers continually proportional* (E, D, and F, G,) *fall as many mean numbers in continnal proportion as fall between either of them and unity, ſo many means alſo ſhall fall in continual proportion between them,* I, K.

For E, DF, G, and A, DqF (I) DG (K) B are ∺ by 2. 8, therefore, &c.

## PROP. XI.

A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.

*Between two ſquare numbers* Aq, Bq, *there is one mean proportional number* AB: *and* Aq *to* Bq, *is in duplicate proportion of the ſide* A *to the ſide* B.

a 17. 7.
b 10. *def.* 5.

(*a*) It is manifeſt that Aq, AB, Bq, are ∺; (*b*) and conſequently alſo $\frac{A\,q}{B\,q} = \frac{A}{B}$ twice. *Which was to be demonſtrated.*

## PROP. XII.

Ac, 27. AqB, 36. BqA, 48. Bc, 64.
A, 3. B, 4.
Aq, 9. AB, 12. Bq, 16.

*Between two cube numbers,* Ac, Bc, *there are two mean proportional numbers* AqB, BqA; *and the cube* Ac *is to the cube* Bc *in triplicate ratio of the ſide* A *to the ſide* B.

a 2. 8.
b 10. *def.* 5.

(*a*) For Ac, AqB, BqA, Bc, are ∺ in the proportion of A to B; (*b*) and therefore $\frac{A\,c}{B\,c} = \frac{A}{B}$ thrice. *Which was to be demonſtrated.*

## PROP. XIII.

A, 2. B, 4. C, 8.
Aq, 4. AB, 8. Bq, 16. BC, 32. Cq, 64.
AC, 8. AqB, 16. BqA, 32. Bc, 64. BqC, 128. CqB, 256. Cc, 512.

*If there be numbers in continual proportion how many ſoever* A, B, C; *and every of them multiplying itſelf produceth certain numbers; the numbers produced of them* Aq, Bq, Cq, *ſhall be proportional: And if the numbers firſt given* A, B, C, *multipling their products* Aq, Bq, Cq, *produce other numbers* Ac, Bc, Cc, *they alſo ſhall be proportional; and this ſhall ever happen to the extremes.*

For Aq, AB, Bq, BC, Cq (*a*) are ∺; (*b*) therefore by equality Aq : Bq : : Bq : Cq. *Which was to be demonſt.*

a 2. 8.
b 14. 7.

(*a*) Alſo Ac, AqB, BqA, Bc, BqC, CqB, Cc, are ∺; (*b*) therefore again by equality Ac : Bc : : Bc : Cc. *Which was to be demonſtrated.*

PROP.

## PROP. XIV.

*If a ſquare number* Aq *meaſure a ſquare number* Bq, *the ſide alſo of the one* (A) *ſhall meaſure the ſide of the other* (B) *: and if the ſide of one ſquare* A, *meaſure the ſide of another* B, *the ſquare* Aq *ſhall likewiſe meaſure the ſquare* Bq.

Aq. 4. AB, 12. Bq, 36.
A, 2. B, 6.

1. *Hyp.* For Aq : AB (*a*) : : AB : Bq, therefore ſeeing by a 2.&11.8
the hypotheſis Aq meaſures Bq, (*b*) it ſhall meaſure alſo b 7. 8.
AB. But Aq : AB : : A : B, (*c*) therefore alſo A meaſures c 20.*def.*7.
B. *Which was to be demonſtrated.*

2. *Hyp.* A meaſures B ; (*c*) therefore Aq ſhall as well
meaſure AB, (*c*) as AB meaſures Bq ; (*d*) conſequently Aq d 11. *ax.* 7.
meaſures Bq. *Which was to be demonſtrated.*

## PROP. XV.

*If a cube number* Ac *meaſures a cube number* Bc, *then the ſide of the one* (A) *ſhall meaſure the ſide of the other* (B.) *And if the ſide* A *of one cube* Ac *meaſure the ſide* B *of the other* Bc, *alſo the cube* Ac *ſhall meaſure the cube* Bc.

A, 2. B, 6.
Ac, 8. AqB, 24. ABq, 72. Bc, 216.

1. *Hyp* For Ac, AqB, BqA, Bc (*a*) are ∺ ; therefore a 2.&12.8
Ac, (*b*) meaſuring the extreme Bc, ſhall alſo (*c*) meaſure the b *hyp.*
ſecond AqB. But Ac : AqB : : A : B, (*d*) therefore A c 7. 7.
ſhall alſo meaſure B. *Which was to be dem.* d 20. *def.*7

2. *Hyp.* A meaſures B ; (*d*) therefore Ac meaſures e 11.*ax.* 7.
AqB, which alſo meaſures BqA, and this Bc ; (*e*) therefore Ac ſhall meaſure Bc. *Which was to be dem.*

## PROP. XVI.

*If a ſquare number* Aq *do not meaſure a ſquare number* Bq, *neither ſhall the ſide of the one* A *meaſure the ſide of the other* B : *And if* A *the ſide of the one ſquare* Aq *do not meaſure* B *the ſide of the other* Bq, *neither ſhall the ſquare* Aq *meaſure the ſquare* Bq.

A, 4. B, 9.
Aq, 16. Bq, 81.

1. *Hyp.* For if you affirm that A meaſures B, then Aq alſo ſhall meaſure Bq. *Againſt the Hypotheſis.*

2. *Hyp.* If you maintain Aq to meaſure Bq ; (*a*) then a 14. 8.
likewiſe A ſhall meaſure B. *Contrary to the Hypotheſis.*

## PROP. XVII.

A, 2. B. 3.
Ac, 8. Bc, 27.

*If a cube number* Ac *does not meaſure a cube number* Bc, *neither ſhall the ſide of one* A *meaſure the ſide of the other* B: *And if* A *the ſide of one cube* Ac *does not meaſure* B *the ſide of the other* Bc, *neither ſhall the cube* Ac *meaſure the cube* Bc.

a 15. 8. 1. *Hyp.* Let A meaſure B; (*a*) then Ac ſhall meaſure Bc. *Againſt the Hypotheſis.*

2. *Hyp.* Let Ac meaſure Bc; then A ſhall meaſure B; *which is alſo againſt the Hypotheſis.*

## PROP. XVIII.

C, 6. D, 2.
CD, 12.
E, 9. F, 3. DE, 18.
EF, 27.

*Between two like plane numbers* CD *and* EF *there is one mean proportional number* DE: *And the plane* CD *is to the plane* EF *in duplicate proportion of that which the ſide* C *hath to the homologous ſide* E.

* 21. *def.* 7. For * by the Hypotheſis C : D : : E : F; therefore by
a 17. 7. permutation, C : E : : D : F. But C : E (*a*) : : CD : DE;
b 11. 5. (*a*) and D : F : : DE : EF; (*b*) therefore CD : DE : :
c 10. *def.* 5. DE : EF. (*c*) Wherefore the proportion of CD to EF is duplicate to that of CD to DE, that is, to the proportion of C to E, or D to F. *Which was to be demonſtrated.*

*Coroll.*

Hence it is apparent, That between two like plane numbers there falls one mean proportional in the proportion of the homologous ſides.

## PROP. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.
CD, 6. DF, 12. FG, 24.
C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

*Between two like ſolid numbers* CDE, FGH, *there are two mean proportional numbers* DFE, FGE. *And the ſolid* CDE *is to the ſolid* FGH, *in triplicats proportion of that which the homologous ſide* C *has to the homologous ſide* F.

Whereas by the * hyp. C : D : : F : G, and D : E : : * 21. *def.* 7.
G : H ; therefore (*a*) by permutation, ſhall C : F : : D : a 13. 7.
G : : (*a*) E : H. But CD : DF (*b*) : : C : F, and DF : FG b 17. 7.
(*b*) : : D : G ; (*c*) wherefore CD : DF : : DF : FG : : E : c 11. 5.
H ; (*d*) therefore CDE : DFE : : DFE : FGE : : E : H d 17. 7.
: : FGE : FGH. Therefore between CDE, FGH, fall e 10. *def.* 5.
two mean proportionals DFE, FGE. (*e*) And ſo it is plain that the proportion of CDE to FGH is triplicate to that of CDE to DFE, or C to F. *Which was to be dem.*

*Coroll.*

Hereby it is manifeſt, that between two like ſolid numbers there fall two mean proportionals in the proportion of the homologous ſides.

## PROP. XX.

*If between two numbers* A, B, *there falls one mean proportional number* C ; *thoſe numbers* A, B, *are like plane numbers.*

A, 12. C, 18. B, 27.
D, 2. E, 3. F, 6. G, 9.

(*a*) Take D and E the leaſt in the proportion of A to C, a 35. 7.
or C to B, then D meaſures A equally as E does C ; ſuppoſe by the ſame number F ; (*b*) alſo D equally meaſures b 21. 7.
C, as E does B, ſuppoſe by the ſame number G. (*c*) There- c 9. *ax.* 7.
fore DF = A, and EG = B ; (*d*) and conſequently A and B d 16. *def.* 7.
are plane numbers. But becauſe EF (*c*) = C (*c*) = DG ; e 19. 7.
(*e*) therefore D : E : : F : G, and alternately D : F : : f 21. *def.* 7.
E : G. (*f*) Therefore the plane numbers A and B are alſo like. *Which was to be dem.*

## PROP. XXI.

*If between two numbers* A, B, *there fall two mean proportional numbers* C, D ; *thoſe numbers* A, B, *are like ſolid numbers.*

A, 16. C, 24. D, 36. B, 54.
E, 4. F, 6. G, 9.
H, 2. P, 2. M, 4. K, 3. L, 3. N, 6.

(*a*) Take E, F, G, the leaſt ∺ in the proportion of A a 2. 8.
to C, (*b*) then D and G are like plane numbers : let the b 10. 8.
ſides of this be H and P, and of that K and L ; (*c*) there- c 21. *def.* 7.
fore H : K : : P : L : : (*d*) E : F But E, F, G, (*e*) equally d *cor.* 18. 8.
meaſure A, C, D, ſuppoſe by the ſame number M, and e 21. 7.
likewiſe the ſaid numbers E, F, G, equally meaſure the numbers C, D, B, ſuppoſe by the ſame number, N.
(*f*) Therefore A = EM = HPM, (*f*) and B = GN = KLN ; f 9. *ax.* 7.
(*g*) and

g 17. *def.* 7, (*g*) and ſo A and B are ſolid numbers. But becauſe C (*f*)
h 17. 7. = FM, and D (*f*) = FN, therefore M : N (*h*) : : FM :
k 7. 5. FN (*k*) : : C : D (*l*) : : E : F : : H : K : : P : L; (*m*)
l *conſt.* therefore A and B are like ſolid numbers. *Which was to*
m 21. *def.* 7. *be demonſtrated.*

*Lemma.*

AE, BF, CG, DH,
A, B, C, D,
E, F, G, H,

*If proportional numbers* A, B, C, D, *meaſure proportional numbers* AE, BF, CG, DH, *by the numbers* E, F, G, H, *theſe numbers* (E, F, G, H,) *ſhall be proportional.*

a 19. 7. For becauſe AEDH (*a*) = BFCG, (*a*) and AD = BC,
b 1. *ax.* 7. (*b*) ſhall $\frac{AEDH}{AD} = \frac{BFCG}{BC}$ (*c*) that is, EH = FG.
c 9. *ax.* 7. (*a*) Therefore E : F : : G : H. *Which was to be demon.*

*Coroll.*

d 15. *def.* 7. Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. (*e*) For 1 : B : : B : Bq, (*d*) and
e *lem. prec.* 1 : A : : A : Aq, (*d*) therefore 1 : $\frac{B}{A}$ : : $\frac{B}{A}$ : $\frac{Bq}{Aq}$, (*d*) therefore $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. In like manner $\frac{B}{Ac} \times \frac{Bq}{Ac} = \frac{Bc}{Acc}$. and ſo of the reſt.

## PROP. XXII.

Aq, B, C.
4, 8, 16,

*If three numbers* Aq, B, C, *are continually proportional, and the firſt* Aq *a ſquare, the third* C *ſhall alſo be a ſquare.*

a 20. 7. For becauſe AqC (*a*) = Bq, (*b*) thence is C = $\frac{Bq}{Aq}$
b 7. *ax.* 7. (*c*) = Q.$\frac{B}{A}$. But it is plain that $\frac{B}{A}$ is a number, (*d*) be-
c *cor. of the lem prec.* cauſe $\frac{Bq}{Aq}$, or C is a number. Therefore if three, &c.
d *hyp. and* 14. 8.

## PROP. XXIII.

Ac, B, C, D,
8, 12, 18, 27.

*If four numbers* A, B, C, D, *are continually proportional, and the firſt of them* Ac *a cube, the fourth alſo* D, *all be a cube.*

For becauſe $Ac\,D$ (*a*) $= BC$, (*b*) therefore $D = \frac{BC}{Ac}$ (*c*) $= \frac{B}{Ac} \times C$; that is, (becauſe $A\,c\,C =$ (*d*) $Bq$, and (*b*) thence $C = \frac{Bq}{Ac}$) $D = \frac{B}{Ac} \times \frac{Bq}{Ac}$ (*c*) $= \frac{Bc}{Acc}$ (*c*) $= C : \frac{B}{Ac}$

a 19. 7.
b 7. *ax.* 7.
c *cor. of the prec. lem.*
d 20. 7.

But it is evident (*e*) that $\frac{B}{Ac}$ is a number, becauſe $\frac{Bc}{Acc}$ or D is ſuppoſed a number. Therefore if four numbers, *&c.* e 15. 8.

## PROP. XXIV.

*If two numbers* A, B, *are in the ſame proportion one to another, that a ſquare number* C *is to a ſquare number* D, *and the firſt* A *be a ſquare number, the ſecond alſo* B *ſhall be a ſquare number.*

A, 16. 24. B, 36.
C, 4. 6. D, 9.

Between C and D the ſquare numbers, * and ſo be- * 8. 8.
tween A and B having the ſame proportion, (*a*) falls one a 11. 8.
mean proportional. Therefore (*b*) ſince A is a ſquare b *hyp.*
number, (*c*) B alſo ſhall be a ſquare number. *Which was* c 22. 8.
*to be demonſtrated.*

*Coroll.*

1. Hence, if there be two like numbers AB, CD, (A : B : : C : D) and the firſt AB be a ſquare, the ſecond alſo CD ſhall be a ſquare.

* For AB : CD : : Aq : Cq. * 11 *and*

2. From hence it appears, That the proportion of any 18. 8.
ſquare number to any other not ſquare, cannot poſſibly be reſolved into two ſquare numbers. Whence it cannot be Q : Q : : 1 : 2, nor 1 : 5 : : Q : Q, *&c.*

## PROP. XXV.

*If two numbers* A, B, *are in the ſame proportion one to another, that a cube number* C *is to a cube number* D, *the firſt of them* A *being a cube number; the ſecond* B *ſhall likewiſe be a cube number.*

C, 64. 96. 144. D, 216.
A, 8. 12. 18. B, 27.

(*a*) Between the cube numbers C and D, (*b*) and ſo be- a 12. 8.
tween A and B having the ſame proportion, fall two mean b 8. 8.
proportionals; therefore (*c*) becauſe A is a cube, (*d*) ſhall c *hyp.*
B be a cube alſo. *Which was to be demonſtrated.* d 23. 8.

*Coroll.*

### *Coroll.*

1. Hence, If there be two numbers ABC, DEF, (A : B : : D : E, and B : C : : E : F;) and the firſt ABC be a cube, the ſecond DEF ſhall be a cube alſo.

* 12. *and* 19. 8. * For ABC : DFF : : Ac : Dc.

2. Hence it is evident, That the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

## P R O P. XXVI.

A, 20. C, 30. B, 45.
D, 4. E, 6. F, 9.

*Like plane numbers* A, B, *are in the ſame proportion one to another, that a ſquare number is in to a ſquare number.*

a 18. 8. Between A and B (*a*) falls one mean proportional num-
b 2. 8. ber C; (*b*) take three numbers D, E, F, the leaſt ∺ in
c *cor.* 2. 8. the proportion of A to C, the extremes D, F, (*c*) ſhall
d 14. 7. be ſquare numbers. But by equality A : B (*d*) : : D : F; therefore A : B : : Q : Q. *Which was to be demonſtrated.*

## P R O P. XXVII.

A, 16. C, 24. D, 36. B, 54.
E, 8. F, 12. G, 18. H, 27.

*Like ſolid numbers* A, B, *are in the ſame proportion one to another, that a cube number is to a cube number.*

a 19. 8. (*a*) Between A and B fall two mean proportional num-
b 2. 8. bers, namely, C and D: (*b*) Take four numbers E, F, G, H, the leaſt ∺ in the proportion of A to C, (*b*) the
c 14. 7. extremes E, H, are cube numbers. But A : B (*c*) : : E : H : : C : C. *Which was to be demonſtrated.*

### *Schol.*

See *Clavius.* 1. From hence is inferred, that no numbers in proportion ſuperparticular, or ſuperbipartitent, or double, or any other manifold proportion not denominated from a ſquare number, are like plane numbers.

2. Likewiſe, that neither any two prime numbers, nor any two numbers prime one to another, not being ſquares, can be like plane numbers.

## *The End of the eighth* BOOK.

THE

# The NINTH BOOK OF EUCLIDE's ELEMENTS.

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## PROPOSITION I.

A, 6. B, 54.
Aq, 36. 108. AB, 324.

*If two like plane numbers* A, B, *multiplying one another, produce a number* AB, *the number produced* AB *ſhall be a ſquare number.*

For A : B (*a*) : : Aq : AB; wherefore ſince one mean pro- a 17. 7.
portional (*b*) falls between A and B, (*c*) likewiſe one mean b 18. 8.
proportional number ſhall fall between Aq and AB : c 8. 8.
therefore ſince the firſt Aq is a ſquare number, (*d*) the d 22. 8.
third AB ſhall be a ſquare number alſo. *Which was to be demonſtrated.*

Or thus, Let *ab*, *cd*, be like plane numbers; name-
ly, $a : b : : c : d$, (*x*) therefore $ad = bc$, and ſo likewiſe x 19. 7.
*abcd* or *adbc*, (*y*) $= adad = Q : ad$. y 1, ax. 7.

## PROP. II.

*If two numbers* A, B, *multiplying one another, produce a ſquare number* AB, *thoſe numbers* A, B, *are like plane numbers.*

A, 6. B, 54.
Aq, 36. AB, 324.

For A : B (*a*) : : Aq : AB; wherefore ſince between a 17. 7.
Aq and AB, (*b*) there falls one mean proportional number, b 11. 8.
(*c*) likewiſe one mean ſhall fall between A and B, (*d*) c 8. 8.
therefore A and B are like planes. *Which was to be de-* d 20. 8.
*monſtrated.*

## PROP. III.

A, 2. Ac, 8. Acc, 64. *If a cube number* Ac *multiplying it ſelf produce a number* Acc, *the number produced* Acc *ſhall be a cube number.*

a 15. *def.* 7. For, 1 : A (*a*) : : A : Aq (*b*) : : Aq : Ac, therefore be-
b 17. 7. tween 1 and Ac fall two mean proportionals. But 1 : Ac
c 8. 8. (*a*) : : Ac : Acc; (*c*) therefore between Ac and Acc, fall
alſo two mean proportionals; and ſo by conſequence,
d 23. 8. ſince Ac is a cube (*d*) Acc ſhall be a cube alſo.
*Which was to be demonſtrated.*

Or thus; *aaa* (Ac) multiplied into it ſelf make *aaaaaa* (Acc;) this is a cube, whoſe ſide is *aa*.

## PROP. IV.

Ac, 8. Bc, 27.
Acc, 64. AcBc, 216.

*If a cube number* Ac *multiplying a cube number* Bc *produce a number* AcBc, *the produced number* AcBc *ſhall be a cube.*

a 17. 7. For Ac : Bc (*a*) : : Acc : AcBc. But between Ac and
b 12. 8. Bc (*b*) two mean proportional numbers fall; (*c*) therefore
c 8. 8. there fall as many between Acc and AcBc. So that where-
d 23. 8. as Acc is a cube number, (*d*) AcBc ſhall be ſuch alſo.
*Which was to be demonſtrated.*

Or thus. AcBc = *aaa bbb* (*ababab*) = C : *ab*.

## PROP. V.

Ac, 8. B, 27.
Acc, 64. AcB, 216.

*If a cube number* Ac *multiplying a number* B *produce a cube number* AcB, *the number multiplied* B *ſhall alſo be a cube.*

a 17. 7. For Acc : AcB (*a*) : : Ac : B. But between Acc and
b 12. 8. AcB (*b*) fall two mean proportionals; (*c*) therefore alſo as
c 8. 8. many ſhall fall between Ac and B, whence Ac being a cube
d 23. 8. number, (*d*) B ſhall be a cube number alſo. *Which was
to be demonſtrated.*

## PROP. VI.

A, 8. Aq, 64. Ac, 512.

*If a number* A *multiplying it ſelf produce* Aq *a cube, that number* A *it ſelf is a cube.*

For becauſe Aq (*a*) is a cube, and AqA (Ac) (*b*) alſo a cube; therefore (*c*) ſhall A be a cube. *Which was to be demonſtrated.* a *hyp.* b 19. *def.* 7. c 5. 9.

## PROP. VII.

A, 6. B, 11. AB, 66.
D, 2. E, 3.

*If a compoſed number* A *multiplying any number* B, *produce a number* AB, *the number produced* AB *ſhall be a ſolid number.*

Since A is a compoſed number, (*a*) ſome other number D meaſures it, conceive by E, (*b*) therefore A = DE: (*c*) whence DEB = AB is a ſolid number, *Which was to be demonſtrated.* a 13. *def.* 7. b 9. *ax.* 7. c 17. *def.*

## PROP. VIII.

1. $a$, 3. $a^2$, 9. $a^3$, 27. $a^4$, 81. $a^5$, 242. $a^6$, 729.

*If from unity there are numbers continually proportional how many ſoever* (1, $a$, $a^2$, $a^3$, $a^4$, *&c.*) *the third number from unity* $a^2$ *is a ſquare number; and ſo are all forward, leaving one between* ($a^4$, $a^5$, $a^8$, *&c.*) *But the fourth* $a^3$ *is a cube number; and ſo are all forward, leaving two between* ($a^6$, $a^9$, *&c.*) *The ſeventh alſo* $a^6$ *is both a cube number and a ſquare; and ſo are all forward, leaving five between* ($a^{12}$, $a^{18}$, *&c.*)

For 1. $a^2 = Q : a$, and $a^4 = aaaa = Q : aa$, and $a^5 = aaaaaa = Q : aaa$, *&c.*

2. $a^3 = aaa = C.\ a$, and $a^6 = aaaaaa = C : aa$, and $aaaaaaaaa = C : aaa$, *&c.*

3. $a^6 = aaaaaa = C : aa = Q : aaa$, therefore, *&c.*

Or according to *Euclide*; Becauſe $1 : a$ (*a*) $: : a : a^2$, (*b*) ſhall $a^2 = Q : a$, therefore ſeeing $a^2$, $a^3$, $a^4$, are ∺ (*c*) the third $a^4$ ſhall be a ſquare number; and ſo likewiſe $a^6$, $a^8$, *&c.* Alſo becauſe $1 : a$ (*a*) $: : a^2 : a^3$, therefore ſhall $a^3$, (*b*) $= a^2 \times a = C : a$, (*d*) therefore the fourth from $a^3$, namely $a^6$, ſhall be likewiſe a cube, *&c.* and conſequently $a^6$ is both a cube and a ſquare number, *&c.* a *hyp.* b 20. 7. c 12. 8. d 23. 8.

## PROP. IX.

1. $a$, 4. $a^2$, 16. $a^3$, 64. $a^4$, 256, &c.
1. $a$, 8. $a^2$, 64. $a^3$, 512. $a^4$, 4096.

*If from unity there are numbers how many ſoever, continually proportional* (1. $a^2$, $a^3$, *&c.*) *and the number following unity,* ($a$) *be a ſquare; then all the reſt,* $a^2$, $a^3$, $a^4$, &c. *ſhall be ſquares alſo. But if the number next unity* ($a$) *be a cube, then all the following numbers* $a^2$, $a^3$, $a^4$, &c. *ſhall be cube numbers.*

1. *Hyp.* For $a^2$, $a^4$, $a^6$, *&c.* are ſquare numbers by the preceding prop. alſo ſince $a$ is taken to be a ſquare,
a 22. 8. ($a$) therefore the third $a^3$ ſhall be a ſquare, and likewiſe $a^5$, $a^7$, *&c.* and ſo all.

b 23. 8. 2. *Hyp.* $a$ is put a cube, ($b$) therefore $a^4$, $a^7$, $a^{10}$, are
c 20. 7. cubes; but by the prec. $a^3$, $a^6$, $a^9$, *&c.* are cubes:
d 3. 9. laſtly, becauſe $1 : a : : a : aa$, ($c$) therefore ſhall $a^2 = Q : a$,
e 23. 8. but a cube multiplied into it ſelf ($d$) produces a cube; therefore $a^2$ is a cube, ($e$) and conſequently the fourth from it $a^5$; and in like manner $a^8$, $a^{11}$, *&c.* are cubes, therefore all, *&c.* *Which was to be demonſtrated.*

More clearly thus. Let $b$ be the ſide of the ſquare number $a$, and ſo the ſeries $a$, $a^2$, $a^3$, $a^4$, *&c* will be otherwiſe expreſſed, thus, $bb$, $b^4$, $b^6$, $b^8$, *&c.* It is evident that all theſe numbers are ſquares, and may be thus expreſſed, $Q : b$, $Q : bb : Q : bbb$, $Q : bbbb$, *&c.*

In like manner, if $b$ be in the ſide of the cube $a$, the ſeries may be expreſſed thus, $b^3$, $b^6$, $b^9$, $b^{12}$, *&c.* or $C : b$, $C : b^2$, $C : b^3$, $C : b^4$, *&c.*

## PROP. X.

1, $a$, $a^2$, $a^3$, $a^4$, $a^5$, $a^6$
1, 2, 4, 8, 16, 32, 64.

*If from unity, there are numbers how many ſoever continually proportional* (1. $a$, $a^2$, $a^3$, *&c.*) *and the number next the unit* ($a$) *be not a ſquare number; then is none of the reſt following a ſquare number excepting* $a^2$ *the third from unity, and ſo all forward, leaving one between* ($a^4$, $a^6$, $a^8$, *&c.*) *But if that* ($a$) *which is next after the unit, be not a cube number, neither is any other of the following numbers a cube, ſaving* $a^3$, *the fourth from the unit, and ſo all forward, leaving two between,* $a^6$, $a^9$, $a^{12}$, *&c.*

1. *Hyp.* For if it be poſſible, let $a^5$ be a ſquare number; therefore becauſe $a : a^2$ (*a*) : : $a^4 : a^5$, and by inverſion, $a^5 : a^4 : : a^2 : a$; and alſo $a^5$ and $a^4$ (*b*) ſquare numbers, and the firſt $a^2$ a ſquare, (*c*) therefore $a$ ſhall be likewiſe a ſquare; *contrary to the hyp.*

a *hyp.*
b *ſuppoſ.* & 8. 9.
c 24. 8.

2. *Hyp.* If it may be, let $a^4$ be a cube; ſince (*d*) by equality $a^4 : a^6 : : a : a^3$, and inverſely $a^6 : a^4 : : a^3 : a$; and alſo ſince $a^6$ and $a^4$ are cubes, and the firſt $a^3$ a cube, (*e*) therefore $a$ ſhall be a cube alſo; *againſt the hypotheſis.*

d 14. 7.
e 25. 8.

### PROP. XI.

1. $a, a^2, a^3, a^4, a^5, a^6$,
1, 3, 9, 27, 81, 243, 729.

*If there are numbers how many ſoever in continual proportion from unity* (1, $a$, $a^2$, $a^3$, *&c.*) *the leſs meaſureth the greater by ſome one of them that are amongſt the proportional numbers.*

Becauſe 1 : $a$ : : $a$ : $aa$ (*a*) therefore $\frac{aa}{a} = a = \frac{aaa}{aa}$ Alſo becauſe 1 : $aa$ (*b*) : : $a$ : $aaa$, therefore (*a*) $\frac{aaa}{a} = aa = \frac{a^4}{aa} = \frac{a^5}{a^3}$, *&c.* Laſtly becauſe 1 : $a^3$ : : (*b*) $a : a^4$, therefore $\frac{a^4}{a^2} = a^2 = \frac{a^6}{a^3}$, &c.

a 5. *ax.* 7. & 20. *def.* 7.
b 14. 7.

### *Coroll.*

Hence, If a number that meaſures any one of proportional numbers, be not one of the ſaid numbers, neither ſhall the number by which it meaſures the ſaid proportional numbers, be one of them.

### PROP. XII.

1, $a$, $a^2$, $a^3$, $a^4$
1, 6, 36, 216, 1296.
B, 3.

*If there are numbers how many ſoever in continual proportion from unity* (1, $a$, $a^2$, $a^3$, $a^4$) *whatſoever prime number* B, *meaſure the laſt* $a^4$, *the ſame* (B) *ſhall alſo meaſure the number* (*a*) *which follow next after unity.*

If you ſay B does not meaſure $a$, (*a*) then B is prime to $a$; (*b*) and therefore B is prime to $a^2$; (*c*) and ſo conſequently to $a^4$, which it is ſuppoſed to meaſure. *Which is abſurd.*

a 31. 7.
b 27. 7.
c 26. 7.

*Coroll.*

1. Therefore every prime number that meaſures the laſt, does alſo meaſure all thoſe other numbers that precede the laſt.

2. If any number not meaſuring that next to unity, does yet meaſure the laſt, it is a compoſed number.

3. If the number next to the unit be a prime, no other prime number ſhall meaſure the laſt.

## PROP. XIII.

1, $a$, $a^2$, $a^3$, $a^4$
1, 5, 25, 125, 625.
H - - G - - F - - E - -

*If from unity there are numbers in continual proportion, how many ſoever* (1, $a$, $a^2$, $a^3$, &c.) *and that after unity* (a) *a prime; then ſhall no other meaſure the greateſt number, but thoſe which are amongſt the ſaid proportional numbers.*

If it be poſſible, let ſome other E meaſure $a^4$, *viz.* by
a *cor.* 12. 9. F, (a) then F ſhall be ſome other different from $a$, $a^2$,
$a^3$. But becauſe E meaſuring $a^4$, does not meaſure $a$,
b 2. *cor.* 12. 9. (b) therefore E ſhall be a compoſed number, (c) therefore
ſome prime number meaſures it, (d) which does conſe-
c 33. 7. quently meaſure $a^4$, (e) and ſo is no other than $a$, there-
d 11. *ax.* 7. fore $a$ meaſures E. After the ſame manner alſo may
e 3. *cor.* 12. & 9. F be ſhewn to be a compoſed number, meaſuring $a^4$,
and ſo that $a$ meaſures F. Therefore ſeeing EF (f) $= a^4$.
f 9. *ax.* 7. $= a \times a^3$, (g) it will be $a : E : : F : a^3$, Conſequently
g 19. 7. whereas, $a$ meaſures E, (h) likewiſe F ſhall equally mea-
h 20. *def.* 7. ſure $a^3$, *viz.* by the ſame number G. (k) Nor ſhall G be
k *cor.* 11 9. $a$, or $a^2$, therefore, as before, G is a compoſed number,
and $a$ meaſures it. Wherefore ſince FG (f) $= a^3 = a^2 \times$
$a$ (g) therefore $a : F : : G : a^2$, and ſo becauſe A meaſures
F, (h) G ſhall equally meaſure $a^2$, *viz.* by the ſame num-
ber H, (k) which is not $a$. Therefore ſince GH $= a^2 =$
l 20. 7. $aa$ (l) thence $H : a : : a : G$, and becauſe $a$ meaſures G
m 20. *def.* 7. (as before (m) H alſo ſhall meaſure $a$, which is a prime
number. *Which is impoſſible.*

PROP.

## PROP. XIV.

*If certain prime numbers* B, C, D, *meaſure the leaſt number* A, *no other prime number* E *ſhall meaſure the ſame beſides thoſe that meaſured it at firſt.*

A, 30.
B, 2. C, 3. D, 5.
E - - F - - -

If it is poſſible, let $\frac{A}{E}$ be = F; (*a*) then A = EF, (*b*) a 9. *ax.* 7.
therefore each of the prime numbers B, C, D, meaſures b 32. 7
one of thoſe E, F. Not E, which is taken to be a prime; therefore F, which is leſs than A it ſelf; *contrary to the hypotheſis.*

## PROP. XV.

*If three numbers continually proportional* A, B, C, *are the leaſt of all that have the ſame proportion with them; any two of them added together ſhall be a prime to the third.*

A, 9. B, 12 C, 16.
D, 3. E, 4.

(*a*) Take D and E the leaſt in proportion of A to B; (*b*) a 35. 7.
then A = Dq, and (*b*) C = Eq, (*b*) and B = DE. But b 2. 8.
becauſe D (*c*) is prime to E, (*d*) therefore ſhall D + E be c 24. 7.
prime to both D and E, * therefore $D \times \overline{D+E}$ (*e*) = Dq d 30. 7.
+ DE (A + B) (*f*) is prime to E, and ſo to C or Eq. * 26. 7.
*Which was to be demonſtrated.* e 3. 2,

(*g*) In like manner DE + Eq (B + C) is prime to D, f *as before*
and conſequently to A = Dq. *Which was to be demonſtrated.* g 27. 7.

Laſtly, becauſe B (*h*) is prime to D + E, it ſhall h 26. 7.
alſo be prime to the ſquare of it (*k*) Dq + 2 DE + Eq k 4. 2.
(A + 2 B + C;) (*l*) wherefore the ſaid B ſhall be prime l 30. 7.
to A + B + C, (*l*) and ſo likewiſe to A + C. *Which was to be demonſtrated.*

## PROP. XVI.

*If two numbers* A, B, *are prime to one another, it ſhall not be as the firſt* A, *to the ſecond* B, *ſo is the ſecond* B *to any other* C.

A, 3, B, 5. C - - - -

a 23. 7. If you affirm A : B : : B : C, then whereas A and B
b 21. 7. (*a*) are the leaſt in their proportion, A (*b*) ſhall meaſure
c 6. *ax*. 7. B as many times as B does C; but A (*c*) meaſures it ſelf alſo; therefore A and B are not prime to one another. *againſt the hypotheſis.*

## PROP. XVII.

A, 8. B, 12. C, 18. D, 27. E - - - *If there are numbers how many ſoever in continual proportion* A, B, C, D, *and the extremes of them* A, D, *be prime one to another, the firſt* A *ſhall not be to the ſecond* B, *as the laſt* D *to any other* E.

Suppoſe A : B : : D : E, then alternately A : D : :
a 23. 7. B : E, therefore ſeeing A and B are (*a*) the leaſt in their
b 21. 7. proportion, A (*b*) ſhall meaſure B, (*c*) and B likewiſe C,
c 20. *def*. 7. and C the following number D; (*d*) and ſo A ſhall mea-
d 11. *ax*. 7. ſure the ſaid number D. Wherefore A and D are not prime to one another; *contrary to the hypotheſis.*

## PROP. XVIII.

A, 4. B, 6. C, 9.
Bq, 36.

*Two numbers being given* A, B, *to conſider if there may be a third number* C *found proportional to them.*

a 9. *ax*. 7. If A meaſures Bq by any number C, (*a*) then AC=Bq,
b 20. 7. from whence (*b*) it is manifeſt that A : B : : B : C. *Which was to be done.*

A, 6. B, 4. Bq, 16.

But if A does not meaſure Bq, there will not be any third proportional. For ſuppoſe A : B
c 7. *ax*. 7. : : B : C; (*a*) then AC=Bq, (*c*) and conſequently $\frac{Bq}{A} = C$, namely A meaſures Bq. *Which is againſt the Hypotheſis.*

## PROP. XIX.

A, 8. B, 12. C, 18. D, 27.
BC, 216.

*Three numbers being given* A, B, C, *to conſider if a fourth proportional to them* D *may be found.*

a 9. *ax*. 7. If A meaſures BC by any number D, (*a*) then AD=
b *ax*. 19. 7. BC; (*b*) therefore it appears that A : B : : C : D, which was required.

But if A does not meaſure BC, then there can no fourth proportional be found; which may be ſhewn as in the laſt *prop.*

## PROP. XX.

*More prime numbers may be given than any multitude whatſoever of prime numbers* A, B, C, *propounded.*

A, 2. B, 3. C, 5.
D, 30. G - - - -

(a) Let D be the leaſt which A, B, C, meaſure; If $D + 1$ be a prime, the caſe is plain: if compoſed, (*b*) then ſome prime number, ſuppoſe G, meaſures $D + 1$, which is none of the three A, B, C; For if it be, ſeeing it (*c*) meaſures the whole $D + 1$, (*d*) and the part taken away D, (*e*) it ſhall alſo meaſure the remaining unit. *Which is abſurd.* Therefore the propounded number of prime numbers is increaſed by $D + 1$, or at leaſt by G.

a 38. 7.
b 33. 7.
c *ſuppoſ.*
d *conſtr.*
e 12. *ax.* 7.

## PROP. XXI.

5 5 3 3 2 2
A.....E.....B...F...C..G..D 20.

*If even numbers, how many ſoever* AB, BC, CD, *are added together, the whole* AD *ſhall be even.*

(*a*) Take $EB = \frac{1}{2} AB$, and $FC = \frac{1}{2} BC$, and $GD = \frac{1}{2} CD$; (*b*) it is plain that $EB + FC + GD = \frac{1}{2} AD$, (*c*) therefore AD is an even number. *Which was to be demonſtrated.*

a 6. *def.* 7.
b 12. 7.
c 6 *def.* 7.

## PROP. XXII.

1 1 1 1
A........F.B.......G.C.....H.D...L.E 22.
9 7 5 3

*If odd numbers, how many ſoever,* AB, BC, CD, DE, *are added together, and the multitude of them be even, the whole alſo* AE *ſhall be even.*

Unity being taken from each odd number, there will (*a*) remain AF, BG, CH, DL, even numbers, (*b*) and thence the number compounded of them will be even, add to them the (*c*) even number made of the remaining units, and the (*d*) whole AE will thereby be even. *Which was to be demonſtrated.*

a 7. *def.* 7.
b 21. 9.
c *hyp.*
d 21. 9.

## PROP. XXIII.

7 5 1
A....... B ..... C. . . E . D 15.
3

*If odd numbers how many ſoever,* AB, BC, CD, *are added together, and the multitude of them be odd, the whole* AD *ſhall be odd.*

For CD, one of the odd numbers, being taken away,
a 22. 9. the aggregate of the others AC (*a*) is even. Whereto
b 21. 9. add CD — 1, (*b*) the whole AE is alſo even; wherefore
c 7. *def.* 7. the unit being reſtored the whole AD (*c*) will be odd.
*Which was to be demonſtrated.*

## PROP. XXIV.

4 5 1
A . . . . B . . . . . D . C 10.
6

*If an even number* AB *be taken away from an even number* AC, *that which remains* BC *ſhall be even.*

a 7. *def.* 7. For if BD (BC—1) be odd, (*a*) BC (BD+1) will be
b *hyp.* even. *Which was to be demonſtrated* But if you ſay BD
c 21. 9. even, becauſe A C (*b*) is even, (*c*) thence AD will be
ſo; (*a*) and conſequently AC (AD+1) will be odd, *contrary the Hypotheſis,* therefore BC is even. *Which was to be demonſtrated.*

## PROP. XXV.

6 1 3
A...... D . C ... B 10.
7

*If from an even number* AB, *an odd number* A C *be taken away, the remaining number* CB *ſhall be odd*

a 7. *def.* 7. For AC—1 (AD) (*a*) is even (*b*) therefore DB is even;
b 24. 9. (*c*) and conſequently CB (DB—1) is odd. *Which was to*
c 7 *def.* 7. *be demonſtrated.*

## PROP. XXVI.

4 6 1
A .... C ...... D . B 11.

*If from an odd number* AB *be taken away an odd number* C B, *that which remaineth* AC *ſhall be even.*

For AB − 1 (AD) and CB−1 (CD) (*a*) are even; (*b*) therefore AD−CD (AC) is even. *Which was to be demonſtrated.*

a 7. *def.* 7.
b 24. 9.

## PROP. XXVII.

*If from an odd number* AB *be taken away an even number* CB, *the reſidue* AC *ſhall be odd.*

1 4 6
A . D . . . . C . . . . . . B 11.
5

For AB−1 (DB) (*a*) is even, and CB is ſuppoſed to be even: (*b*) therefore the reſidue CD is even: (*c*) therefore CD+1 (CA) is odd, *Which was to be demonſtrated.*

a 7. *def.* 7.
b 24. 9.
c 7. *def.* 7.

## PROP. XXVIII.

*If an odd number* A, *multiplying an even number* B, *produces a number* AB, *the number produced* AB *ſhall be even.*

A, 3
B, 4
AB, 12.

For AB (*a*) is compounded of the odd number A taken as many times as an unit is contained in B, an even number. (*b*) Therefore AB is an even number. *W.W.D.*

a *hyp. and* 15. *def.* 7.
b 21. 9.

*Schol.*

In like manner, if A be an even number, AB ſhall be an even number alſo.

## PROP. XXIX.

*If an odd number* A *multiplying an odd number* B, *produces a number* AB, *the number produced* AB, *ſhall be odd.*

A, 3.
B, 5.
AB, 15.

For AB (*a*) is compounded of the odd number B taken as often as an unit is included in A, likewiſe an odd number. (*b*) Therefore AB is an odd number. *Which was to be demonſtrated.*

a 15. *def.* 7.
b 23. 9.

*Schol.*

1. *An odd number* A *meaſuring an even number* B, *meaſures the ſame by an even number* C.

B, 12. (C, 4
A, 3.

For if C be affirmed to be odd, then becauſe (*a*) B = AC, (*b*) therefore B ſhall be odd; *againſt the hyp.*

a 9 *ax.* 7.
b 29. 9.

2. *An*

$\frac{\text{B, 15.}}{\text{A, 3.}}$ (C, 5.

2. *An odd number* A *meaſuring an odd number* B, *meaſures the ſame by an odd number* C.

a 28. 9. For if C be ſaid to be even, (*a*) then AC, or B will be even, *contrary to the Hypotheſis.*

$\frac{\text{B, 15.}}{\text{A, 3.}}$ (C, 5.

3. *Every number* (A *and* C) *that meaſures an odd number* B, *is it ſelf an odd number.*

a 28. 9. For if either A or C be affirmed to be even, B (*a*) ſhall be an even number, *againſt the Hypotheſis.*

## PROP. XXX.

$\frac{\text{B, 24.}}{\text{A, 3.}}$ (C, 8. $\frac{\text{D, 12.}}{\text{A, 3.}}$ (E, 4.

*If an odd number* A *meaſures an even number* B, *it ſhall alſo meaſure the half of it* D.

a hyp.
b 1 ſchol. 29. 9.
c 9. ax. 7.
d 1. 2.
e hyp.
f 7. ax. 1.
g 7. ax. 7.

(*a*) Let $\frac{B}{A}$ be $=$ C, (*b*) then C is an even number. Therefore let E be $= \frac{1}{2}$ C, then B (*c*) $=$ CA (*d*) $= 2$ EA (*e*) $= 2$ D, (*f*) therefore EA $=$ D; (*g*) and conſequently $\frac{D}{A} =$ E. *Which was to be demonſtrated.*

## PROP. XXXI.

A, 5. B, 8. C, 16. D - - - *If an odd number* A *be prime to any number* B, *it ſhall alſo be prime to the double thereof* C.

a 3. ſchol. 29. 9.
b 30. 9.

If it be poſſible, let ſome number D meaſure A and C, (*a*) then D meaſuring the odd number A ſhall be odd it ſelf, (*b*) and ſo ſhall meaſure B, the half of the even number C, therefore A and B are not prime one to another. *Which is againſt the Hypotheſis.*

### *Coroll.*

It follows from hence, that an odd number which is prime to any number of a double progreſſion, is alſo prime to all the numbers of that progreſſion.

## PROP. XXXII.

*All numbers* A, B, C, D, &c. *in double progreſſion from two, are evenly even only.*

1. A, 2. B, 4. C, 8. D, 16.

It is evident that all theſe numbers A, B, C, D, (*a*) are a 6. *def.* 7.
even, and (*b*) ∺, namely in a double proportion, (*c*) and b20. *def.* 7.
ſo every leſs meaſures the greater by ſome one of them. c 11. 9.
(*d*) Wherefore all are evenly even. But becauſe A is a d 8. *def.* 7.
prime number, (*e*) no number beſides theſe ſhall meaſure e 13. 9.
any of them. Therefore they are evenly even only. *Which was to be demonſtrated.*

## PROP. XXXIII.

*If of a number* A, *the half of* B *be odd, the ſame* A *is evenly odd only.*

A, 30. B, 15.
D - - - E - -

Since an odd number B (*a*) meaſures A by 2 an a *hyp.*
even number, (*b*) therefore B is evenly odd. If you b 9. *def.* 7.
affirm it to be evenly even, (*c*) then ſome even number c 8. *def.* 7.
D meaſures it by an even number E, whence 2 B (*d*) = d 9. *ax.* 7.
A (*d*) = DE; (*e*) therefore 2 : E : : D : B, and there- e 19. 7.
fore as 2 (*f*) meaſures the even number E, (*g*) ſo D an f 6. *def.* 7.
even number meaſures B an odd. *Which is impoſſible.* g 20. *def.* 7.

## PROP. XXXIV.

*If an even number* A *be neither doubled from two, nor have it's half odd, it is both evenly even and evenly odd.*

A, 24.

It is evident that A is evenly even, becauſe the half of it is not odd. But becauſe, if A be divided into two equal parts, and the half of it be again divided into two equal parts, and if this be always done, we ſhall at length
fall upon ſome (*a*) odd number, (for we cannot fall upon a 7. *def.* 7.
the number two, becauſe A is not ſuppoſed to be doubled upward from two) which ſhall meaſure A by an even
number, for (*b*) otherwiſe A it ſelf ſhould be odd; *againſt* b 1. *ſch.* 29.
*the Hyp.*. Therefore A is evenly odd. *Which was to be* 9.
*demonſtrated.*

## PROP. XXXV.

A . . . . . . . . 8.
4 8
B . . . . F . . . . . . . . G 12.
C . . . . . . . . . . . . . . . . . . 18.
9 6 4 8
D . . . . . . . . . H . . . . . . L . . . . K . . . . . . . . . N 27.

*If there are numbers in continual proportion how many ſoever* A, BG, C, DN, *and the number* FG, *equal to the firſt* A, *be taken from the ſecond, and* KN, *alſo equal to the firſt, from the laſt; it ſhall be as the exceſs of the ſecond* BF *is to the firſt* A, *ſo is the exceſs of the laſt* DK *to all the numbers that precede it,* A, BG, C.

From DN take NL = BG, and NH = C. Becauſe
a *hyp.* DN : C (HN) (*a*) : : HN : BG (LN) (*a*) : : LN (BG) : A
b 17. 5. (KN ;) (*b*) therefore by dividing every where, it will be
c 12. 5. DH : HN : : HL : LN : : LK : KN ; (*c*) wherefore DK :
d 3. *ax*. 1. C + BG + A : : LK (*d*) (BF) : KN (A.) *Which was to be demonſtrated.*

*Coroll.*

e 18. 5. Hence (*e*) by compounding, DN + BG + C : A + BG + C : : BG : A.

## PROP. XXXVI.

1. A, 2. B, 4. C, 8. D, 16.
E, 31. G, 62. H, 124. L, 248. F, 496.
M, 31. N, 465.
P - - - - Q - - -

*If from unity be taken how many numbers ſoever* 1, A, B, C, D, *in double proportion continually, until the whole added together* E *be a prime number; and if this whole* E *multiply'd into the laſt* D, *produce a number* F, *that which is produced* F, *ſhall be a perfect number.*

Take as many numbers E, G, H, L, likewiſe in double proportion continually; then (*a*) by equality A : D : :
a 14. 7. E : L ; (*b*) therefore AL = DE (*c*) = F, (*d*) whence L =
b 19. 7. $\frac{F}{2}$
c *hyp* Wherefore E, G, H, L, F, are ∺ in double pro-
d 7. *ax*. 7. portion. Let G — E be = M, and F — E = N ; (*e*)
e 35. 9. then M : E : : N : E + G + H + L. (*f*) But M = E, (*g*)
f 3. *ax*. 1. there-
g 14. 5.

therefore $N=E+G+H+L$; (*h*) therefore $F=1+A+B+C+D+E+G+H+L=E+N$. Moreover becauſe D (*k*) meaſures DE (F) (*l*) therefore every one, 1, A, B, C, (*m*) meaſuring D, and (*m*) alſo E, G, H, L, meaſures F. And further, no other number meaſures the ſaid F. For if there does, let it be P, which let meaſures F by Q, (*n*) therefore $PQ=F=DE$; (*o*) therefore $E:Q::P:D$, therefore ſeeing A a prime number, meaſures D, (*p*) and ſo no other P meaſures the ſame, (*q*) conſequently E does not meaſure Q. Wherefore ſince E is ſuppoſed a prime number, (*r*) it ſhall be prime to Q, (*ſ*) wherefore E and Q are the leaſt in their proportion; (*t*) and ſo E meaſures P as many times as Q does D: (*u*) therefore Q is one of them A, B, C. Let it be B, ſeeing then by equality $B:D::E:H$, (*x*) and ſo $BH=DE=F=PQ$, (*x*) and ſo alſo $Q:B::H:P$, (*y*) therefore $H=P$; therefore P is alſo one of them A, B, C, &c. *Againſt the Hypotheſis.* Therefore no other beſide the foreſaid numbers meaſure F, and (*z*) conſequently F is a perfect number, *Which was to be demonſtrated.*

h 2. *ax.* 1.
k 7. *ax.* 7.
l 11. *ax.* 7.
m 11. 9.
n 9. *ax.* 7.
o 19. 7.
p 13. 9.
q 20. *def.* 7.
r 31. 7.
ſ 23. 7.
t 21. 7.
u 13. 9.
x 19. 7.
y 14. 5.
z 22. *def.* 7.

*The End of the ninth* BOOK.

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The

# The TENTH BOOK OF EUCLIDE's ELEMENTS.

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## *Definitions.*

I. MAgnitudes are ſaid to be commenſurable which one and the ſame meaſure, meaſures them all.

Plate IV. Fig. 10. *The note of commenſurability is* ⊓, *as* A ⊓ B; *that is, the line* A *of* 8 *foot is commenſurable to the line* B *of* 13 *foot; becauſe* D *a line of one foot meaſures both* A *and* B. *Alſo* $\sqrt{18}$ ⊓ $\sqrt{50}$, *becauſe* $\sqrt{2}$ *meaſures both* $\sqrt{18}$ *and* $\sqrt{50}$. *For* $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$ *and* $\sqrt{\frac{50}{2}} = \sqrt{25} = 5$, *wherefore* $\sqrt{18} : \sqrt{50} :: 3 : 5$.

II. Incommenſurable magnitudes are ſuch, of which no common meaſure can be found.

*Incommenſurability is denoted by this mark* ⸢⊓; *as* $\sqrt{6}$ ⸢⊓ $\sqrt{25}$ (5;) *that is,* $\sqrt{6}$ *is incommenſurable to the number* 5, *or to a magnitude deſigned by that number; becauſe there is no common meaſure of them, as ſhall appear hereafter.*

III. Right-lines are commenſurable in power, when the ſame ſpace meaſures their ſquares.

Fig. 11. *The mark of this commenſurability is* ⊓̢; *as* AB ⊓̢ CD, *i. e. the line* AB *of* 6 *foot is in power commenſurable to the line* CD, *which is expreſſed by* $\sqrt{20}$, *becauſe* E *the ſpace of one foot ſquare does as well meaſure* ABq (36) *as the rectangle* XY (20) *to which the ſquare of the line* CD ($\sqrt{20}$) *is equal. The ſame note* ⸢⊓̢ *ſometimes ſignifies commenſurable in power only.*

IV. Lines incommenſurable in power are ſuch, to whoſe ſquares no ſpace can be found to be a common meaſure.

*This incommenſurability is denoted thus ;* 5 ⊓̡ *v* √8. *i. e. the numbers or lines* 5, *and v* √ 8 *are incommenſurable in power, becauſe their ſquares* 25 *and* √ 8 *are incommenſurable.*

V. From which it is manifeſt, that to any right-line given, right-lines infinite in number are both commenſurable and incommenſurable ; ſome in length and power, others in power only. The right-line given is called a Rational-line.

*The note of which is* ρ̊.

VI. And lines commenſurable to this line, whether in length and power, or in power only, are alſo called Rational ρ̊.

VII. But ſuch as are incommenſurable to it, are called Irrational,

*And denoted thus* ῤ.

VIII. Alſo the ſquare which is made of the ſaid given right-line is called Rational, ρ̊ν.

IX. And likewiſe ſuch figures as are commenſurable to it, are Rational ρ̊α.

X. But ſuch as are incommenſurable, Irrational ῤα.

XI. And thoſe right-lines alſo, which contain them in power, are Irrational ρ'.

### *Schol.* Plate IV. *Fig.* 12.

*That the laſt ſeven definitions may be rendered more clear by an example, let there be a circle* ADBP, *whoſe ſemidiameter is* CB, *inſcribe therein the ſides of the ordinate figures, as of a Hexagon* BP, *of a triangle* AP, *of a ſquare* BD, *of a Pentagone* FD. *Therefore, if according to the* 5. def *the ſemidiameter* CB *be the Rational line given, expreſſed by the number* 2, *to which the other lines* BP, AP, BD, FD, *are to be compared, then* BP (a) = BC = 2, *wherefore* BP *is* ρ̊ ⊓ BC, *according to the* 6. def. *Alſo* AP (b) = √ 12 (*for* ABq (16) — BPq (4) = 12) *therefore* AP *is* ρ̊ ⊓̡ BC, *likewiſe according to the* 6. def. *and* APq (12) *is* ρ̊α *by the* 9. def. *Moreover* BD (b) = √ DCq + BCq = 8 ; *whence* BD *is* ρ̊ ⊓̡ BC ; *and* BDq ρ̊ν. *Laſtly,* FDq = 10 — √20 (*as ſhall appear by the praxis to be delivered at the* 10. 13.) *ſhall be* ῤα, *according to the* 10. def. *and conſequently* FD = √ : 10 — √ 20 *is* ρ̊, *according to the* 11. def.

a *cor.* 15. 4.
b 47. 1.

*A Poſtulate*

### *A Poſtulate.*

That any magnitude may be ſo often multiplyed, till it exceed any magnitude whatſoever of the ſame kind.

### *Axioms.*

1. A magnitude meaſuring how many magnitudes ſoever, does alſo meaſure that which is compoſed of them.

2. A magnitude meaſuring any magnitude whatſoever, does likewiſe meaſure every magnitude which that meaſures.

3. A magnitude meaſuring a whole magnitude and a part of it taken away, does alſo meaſure the reſidue.

## PROP. I. Plate IV. *Fig.* 13.

*Two unequal magnitudes* AB, C, *being given, if from the greater* AB *there be taken away more than half* (AH) *and from the reſidue* (HB) *be again taken more than half* (HI) *and this be done continually, there ſhall at length be left a certain magnitude* I B, *leſs than the leſſer of the magnitudes firſt given* C.

a *poſt.* 10. (*a*) Take C ſo often, till its multiple does ſomewhat exceed AB, and let DF = FG = GE = C. Take from AB more than half AH, and from the remainder HB, more than half, *viz.* HI, and ſo continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE which is not leſs than $\frac{1}{2}$ DE, is greater than H B, which is leſs than $\frac{1}{2}$ A B $\sqsupset$ D E. And in like manner GE, which is not leſs than $\frac{1}{2}$ FE, is greater than IB $\sqsupset$ $\frac{1}{2}$ HB, therefore C, or GE $\sqsubset$ IB. *Which was to be demonſtrated.*

The ſame may alſo be demonſtrated, if from AB the half AH be taken away, and again from the reſidue HB the half HI, and ſo forward.

## PROP. II. *Fig.* 14.

*Two unequal magnitudes being given* (AB, CD) *if the leſs* AB *be continually taken from the greater* CD, *by an interchangeable ſubtraction, and the reſidue do not meaſure the magnitude going before, then are the magnitudes given incommenſurable.*

If it be poſſible, let ſome magnitude E be the common meaſure. Then becauſe AB taken from CD, as often as it can be, leaves a magnitude FD leſs than it ſelf, and FD taken from AB leaves GB, and ſo forward (*a*) there- a 1. 10.
fore at length ſome magnitude GB ⊐ E ſhall be left, b *hyp.*
therefore E (*b*) meaſuring AB, (*c*) and ſo CF, (*b*) and c 2. *ax.* 10.
the whole CD, (*d*) ſhall alſo meaſure the reſidue FD, (*c*) d 3 *ax.* 10.
conſequently alſo A G; (*d*) wherefore it ſhall likewiſe meaſure the remainder GB, leſs than it ſelf. *Which is abſurd.*

### PROP. III. Plate IV. *Fig.* 15.

*Two commenſurable magnitudes being given* AB, CD, *to find out their greateſt common meaſure* FB.

Take AB from CD, and the reſidue ED from AB, and FB from ED, till FB meaſure ED (which will come to paſs at length, (*a*) becauſe by the Hyp. AB ⊓⊔ CD) a 2. 10.
FB ſhall be the magnitude required.

For FB (*b*) meaſures ED, (*c*) and ſo alſo AF; but b *conſt.*
it meaſures it ſelf too, (*d*) therefore likewiſe AB, (*c*) c 2. *ax.* 10.
and conſequently CE, (*d*) and ſo the whole CD. Where- d 1. *ax.* 10.
fore FB is the common meaſure of AB, CD. If you affirm G to be a common meaſure greater than that, then G meaſuring AB and CD, (*e*) meaſures alſo CE e 2. *ax.* 10.
and (*f*) the remainder ED, (*e*) and ſo A F; and (*f*) f 3. *ax.* 10.
conſequently the remainder FB, the greater the leſs. *Which is abſurd.*

### *Coroll.*

Hence, a magnitude that meaſures two magnitudes, does alſo meaſure their greateſt common meaſure.

### PROP. IV. *Fig.* 16.

*Three commenſurable magnitudes being given,* A, B, C, *to find out their greateſt common meaſure.*

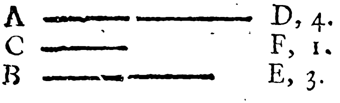
(*a*) Find out D the greateſt common meaſure of any two a 3. 10.
A, B; (*a*) alſo E the greateſt common meaſure of D and 2 *ax.* 10.
C, therefore E is the magnitude ſought.

(*a*) For it is clear, E meaſuring D and C, (*b*) mea-
ſures the three, A, B, C. Conceive another magnitude b *conſtr.* &
F greater than that to meaſure them; (*c*) then F mea- c *cor.* 3. 10.
ſures D, (*c*) and conſequently E the greateſt common meaſure of D, and C, the greater the leſs. *Which is abſurd.*

*Coroll.*

Hence alſo it appears, that if a magnitude meaſures three magnitudes, it ſhall likewiſe meaſure their greateſt common meaſure.

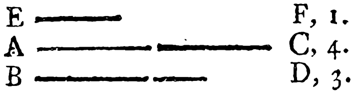
## PROP. V.



*Commenſurable magnitudes* A, B, *have ſuch proportion one to another as number hath to number.*

a 3. 10. (*a*) C being found the greateſt common meaſure of A, B; as often as C is contained in A and B, ſo often is
b 20. *def.* 7. 1 contained in the numbers D and E; (*b*) therefore C : A : : 1 : D; wherefore inverſely A : C : : D : 1, (*b*) but
c 22. 5. likewiſe C : B : : 1. E; (*c*) therefore by equality A : B : : D : E : : N : N. *Note*, The letter N only ſignifies number in general, and refers not to any particular ſpace or magnitude as the other letters do, and is to be read, as A to B, ſo D to E, and ſo number to number. *Which was to be demonſtrated.*

## PROP. VI.



*If two magnitudes* A, B, *have ſuch proportion one to another, as the number* C *hath to the number* D, *thoſe magnitudes* A, B, *ſhall be commenſurable.*

a *ſch.* 10. 6. What part 1 is of the number C, (*a*) that let E be of
b *conſtr.* A. Therefore becauſe E : A (*b*) : : 1 : C, and A : B (*c*)
c *hyp.* : : C : D, (*d*) therefore by equality ſhall E : B : : 1 : D.
d 22. 5. Wherefore ſeeing 1 (*e*) meaſures the number D, (*f*) like-
e 5. *ax.* 7. wiſe E meaſures B; but it (*g*) alſo meaſures A, (*h*) therefore
f 20. *def.* 7. A ⊓ B. *Which was to be demonſtrated.*
g *conſt.*
h 1. *def.* 10.

## PROP. VII. Plate IV. *Fig.* 17.

*Incommenſurable magnitudes* A, B, *have not that proportion one to another, which number hath to number.*

a 6. 10. If you affirm A : B : : N : N, (*a*) then A ⊓ B, *againſt the Hypotheſis.*

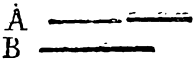
PROP.

## PROP. VIII. Plate IV. *Fig.* 17.

*If two magnitudes* A, B, *have not that proportion one to another, which number hath to number, thoſe magnitudes are incommenſurable.*

Conceive A ⌐⊔ B. (*a*) then A : B : : N : N, *contrary to the Hypotheſis.* a 5. 10.

## PROP. IX.



E, 4.
F, 3.

*The ſquares deſcribed upon right lines commenſurable in length, have that proportion one to another, that a ſquare number hath to a ſquare number. And ſquares, which have that proportion one to another, that a ſquare number hath to a ſquare number, ſhall alſo have their ſides commenſurable in length. But ſuch ſquares as are made upon right-lines incommenſurable in length, have not that proportion one to another, which a ſquare number hath to a ſquare number. And ſquares which have not ſuch proportion one to another, as a ſquare number hath to a ſquare number, have not their ſides commenſurable in length.*

1. *Hyp.* A ⌐⊔ B. I ſay that Aq : Bq : : Q : Q.

For (*a*) let A : B : : number E : number F; therefore $\frac{Aq}{Bq}$ (*b*) $= \frac{A}{B}$ twice (*c*) $= \frac{E}{F}$ twice, (*d*) $= \frac{Eq}{Fq}$, (*e*) therefore Aq : Bq : : Eq : Fq : : Q : Q. *Which was to be demonſtrated.* a 5. 10. b 20. 6. c *ſch.* 23. 5. d 11. 8. e 11. 5.

2. *Hyp.* Aq : Bq : : Eq : Fq : : Q : Q. I ſay A ⌐⊔ B. For $\frac{A}{B}$ twice (*f*) $\frac{Aq}{Bq}$ (*g*) $= \frac{Eq}{Fq}$ (*h*) $= \frac{E}{F}$ twice, (*i*) therefore A : B : : E : F : : N : N, (*k*) wherefore A ⌐⊔ B. *Which was to be demonſtrated.* f 20. 6. g *hyp.* h 11. 8. i *ſch.* 23. 5. k 6. 10.

3. *Hyp.* A ⌙⊔ B I deny that Aq : Bq : : Q : Q. For ſuppoſe Aq : Bq : : Q : Q, then A ⌐⊔ B, as is ſhewn before, *againſt the Hypotheſis.*

4. *Hyp.* Not Aq : Bq : : Q : Q, I ſay that A ⌙⊔ B. For conceive A ⌐⊔ B, then Aq : Bq : : Q : Q, as above, *againſt the Hypotheſis.*

### *Coroll.*

Lines ⌐⊔ are alſo ⌐⊓, but not on the contrary. And lines ⌙⊔ are not therefore ⌐⊓, but Lines ⌐⊓ are alſo ⌙⊔.

### PROP. X. Plate IV. *Fig.* 18.

*If four magnitudes are proportional* (C : A : : B : D) *and the firſt* C *be commenſurable to the ſecond* A, *the third* B *ſhall be commenſurable to the fourth* D. *And if the firſt* C *be incommenſurable to the ſecond* A, *alſo the third* B *ſhall be incommenſurable to the fourth* D.

a 5. 10. If C ⌐⊔ A, (*a*) then C : A : : N : N, (*b*) : : B : D,
b 6. 10. (*b*) therefore B ⌐⊔ D. But if C ⌙⊔ A, (*c*) then ſhall
c 7. 10. not C : A : : N : N : : B : D, (*e*) wherefore B ⌙⊔ D.
d 8. 10. *Which was to be demonſtrated.*

### *Lemma* 1.

*To find out two plane numbers, not having the proportion which a ſquare number hath to a ſquare.*

Any two plane numbers not like, will ſatisfy this Lemma, as thoſe numbers which have ſuper-particular, ſuperbipartient, or double proportion; or any two prime numbers, *See Schol.* 27. 8.

### *Lemma* 2. *Fig.* 19.

*To find out a line* HR, *to which a right-line given* KM *hath the proportion of two numbers given* B, C.

a *ſch* 10. 6. (*a*) Divide KM into as many equal parts as there are
units in the number B, and let as many of theſe, as there
b 3. 1. are units in the number C, (*b*) make the right-line HR,
it is manifeſt that KM : HR : : B : C.

### *Lemma* 3.

*To find out a line* D, *to the ſquare of which the ſquare of a right-line given* KM *hath the proportion of two numbers given* B, C.

a 2. *lem.* 10. Make B : C (*a*) : : KM : HR, and between KM and
10. HR, (*b*) find a mean proportional D. Therefore KMq :
b 13. 6. Dq (*c*) : : KM : HR (*d*) : : B : C.
c 20. 6.
d *conſtr.*

### PROP. XI. *Fig.* 20.

a 1. *lem.* 10. *To find two right-lines incommenſurable to a right-line*
10. *given* A, *one* D *in length only, the other* E *in power alſo.*
b 3. *lem.* 10. 1. Take the numbers B, C, (*a*) ſo that it be not B : C
10. : : Q : Q, (*b*) and let B : C : : Aq : Dq, (*c*) it is plain that
c 9. 10. A ⌙⊔ D. But Aq (*d*) ⌐⊔ Dq. *Which was to be done.*
d 6. 10.

2. (*d*) Make A : E : : E : D. I ſay Aq ˥⊔ Eq. For d 13. 6.
A : D (*e*) : : Aq : Eq, therefore ſince A ˥⊔ D, as before; e 20. 6.
(*f*) therefore Aq ˥⊔ Eq. *Which was to be done.* f 10. 10.

## PROP. XII. Plate IV. *Fig.* 20.

*Magnitudes* (A,B) *commenſurable to the ſame magnitude* C, *are alſo commenſurable one to the other.*

Becauſe A ˥⊔ C, and C ˥⊔ B, (*a*) let A : C : : M : N a 5. 10.
: : D : E, and C : B : : M : N : :
F : G, (*b*) take three numbers H, b 4. 8.
I, K, the leaſt ∺ in the proportion
of D to E, and F to G. Now becauſe
A : C (*c*) : : D : E (*c*) : : H : I, and C : B (*c*) : : F : G c *conſtr.*
: : I : K, (*d*) therefore by equality, A : B : : H : K : : d 22. 5.
M : N, (*e*) therefore A ˥⊔ B. *Which was to be demonſtrated.* e 6. 10.

D, 8. E, 8.
F, 2. G, 3.
H, 5. I, 4. K, 6.

*Schol.*

Hence, every right-line commenſurable to a rational 12. 10. &
line is alſo it ſelf ρ rational. And all rational right-lines *def.* 6.
are commenſurable to one another, at leaſt in power.
Alſo every ſpace commenſurable to a rational ſpace is *def.* 9.
rational too: And all rational ſpaces are commenſurable
one to another. But magnitudes whereof one is rati- *def.* 7. *and*
onal, the other irrational, are incommenſurable amongſt 10.
themſelves.

## PROP. XIII. *Fig.* 21.

*If there are two magnitudes* A, B, *and one of them* A, *commenſurable to a third* C, *but the other* B *incommenſurable, thoſe magnitudes* AB, *are incommenſurable.*

Conceive B ˥⊔ A, then ſince C (*a*) ˥⊔ A, (*b*) there- a *hyp.*
fore C ˥⊔ B, *againſt the Hypotheſis.* b 12. 10.

## PROP. XIV. *Fig.* 20.

*If there are two magnitudes commenſurable* A, B, *and one of them* A *incommenſurable to any other magnitude* C, *the other alſo* B *ſhall be incommenſurable to the ſame* C.

Imagine B ˥⊔ C, then for that A (*a*) ˥⊔ B, (*b*) there- a *hyp.*
fore A ˥⊔ C, *againſt the Hyp.* b 12 10.

## PROP. XV. Plate IV. *Fig.* 22.

*If four right-lines are proportional* (A : B : : C : D) *and the firſt* A *be in power more than the ſecond* B *by the ſquare of a right-line commenſurable to it ſelf in length, then alſo the third* C, *ſhall be more in power than the fourth* D *by the ſquare of a right-line commenſurable to it ſelf in length. But if the firſt* A *be more in power than the ſecond* B, *by the ſquare of a right-line incommenſurable to it ſelf in length, then ſhall the third* C *be more in power than the fourth* D *by the ſquare of a right-line incommenſurable to it ſelf in length.*

a *hyp.* For becauſe A : B (*a*) : : C : D, (*b*) therefore Aq : Bq
b 22. 6. : : Cq : Dq, (*c*) therefore by diviſion Aq — Bq : Bq : :
c 17. 5. Cq — Dq : Dq ; (*d*) wherefore $\sqrt{Aq - Bq}$ : B : : $\sqrt{Cq - Dq}$ : D,
d 22. 6. (*e*) and ſo inverſely B : $\sqrt{Aq - Bq}$ : :
e *cor.* 4. 5. D : $\sqrt{Cq - Dq}$, (*f*) therefore by equality A : $\sqrt{Aq - Bq}$ : : C : $\sqrt{Cq - Dq}$,
f 22. 5. conſequently if A ⊤⊔, or
g 10. 10. ⊥⊔ $\sqrt{Aq - Bq}$, (*g*) then likewiſe C ⊤⊔, or ⊥⊔ $\sqrt{Cq - Dq}$. *Which was to be demonſtrated.*

## PROP. XVI. *Fig.* 23.

*If two commenſurable magnitudes* AB, BC, *are compoſed, the whole magnitude* AC *ſhall be commenſurable to each of the parts* AB, BC. *And if the whole magnitude* AC *be commenſurable to either of the parts* AB, *or* BC, *theſe two magnitudes given at firſt* AB, BC, *ſhall be commenſurable.*

a 3. 10. 1. *Hyp.* (*a*) Let D be the common meaſure of AB,
b 1. *ax.* 10. BC ; (*b*) therefore D meaſures AC, and therefore AC ⊤⊔
c 1. *def.* 10. AB, and BC. *Which was to be demonſtrated.*

2. *Hyp.* (*a*) Let D be the common meaſure of AC, AB,
d 3. *ax.* 10. (*d*) therefore D meaſures AC — AB (BC) and conſequently AB ⊤⊔ BC. *Which was to be demonſtrated.*

### *Coroll.*

Hence it follows, if a whole magnitude compoſed of two, be commenſurable to any one of them, the ſame ſhall be commenſurable to the other alſo.

## PROP. XVII. Plate IV. *Fig.* 23.

*If two incommenſurable magnitudes* AB, BC, *are compoſed, the whole magnitude alſo* AC *ſhall be incommenſurable to either of the two parts* AB, BC. *And if the whole magnitude* AC *be incommenſurable to one of them* AB, *the magnitudes firſt given* AB, BC, *ſhall be incommenſurable.*

1. *Hyp.* If it can be, let D be the common meaſure of AC, AB, (*a*) therefore D meaſures $\overline{AC - AB}$ (BC) (*b*) and therefore alſo AB $\sqcup$ BC, *againſt the hypotheſis.* a 3. *ax.* 10. b 1. *def.* 10. c 16. 10.

2. *Hyp.* Conceive AB $\sqcup$ BC, (*c*) therefore AC $\sqcup$ AB, *againſt the hypotheſis.*

### *Coroll.*

Hence alſo, if one magnitude, compoſed of two, be incommenſurable to any one of them, the ſame alſo ſhall be incommenſurable to the other.

## PROP. XVIII. *Fig.* 24.

*If there are two unequal right-lines* AB, GK, *and upon the greater* AB *a parallelogram* ADB *equal to the fourth part of a ſquare made of the leſs line* GK, *and deficient in figure by a ſquare, be applied, and divides the ſaid* AB *into parts commenſurable in length* AD, DB; *then ſhall the greater line* AB *be more in power than the leſs* GK *by the ſquare of a right-line* FD *commenſurable in length to the greater. And if the greater* AB *be in power more than the leſs* GK, *by the ſquare of the right-line* FD *commenſurable in length to it ſelf, and a parallelogram* ADB *equal to the fourth part of the ſquare made of the leſs line* GK, *and deficient in figure by a ſquare, be applied to the greater* AB, *then ſhall it divide the ſame into parts* AD, DB, *commenſurable in length.*

(*a*) Divide GK equally in H, and (*b*) make the rectangle ADB=GHq. Cut off AF=DB, then is ABq (*c*)= 4 ADB (*d*) (4 GHq or GKq) + FDq. Now in the firſt place, if AD $\sqcup$ DB, then ſhall AB (*e*) $\sqcup$ BD (*e*) $\sqcup$ 2 DB (*f*) (AF+DB, or AB − FD) (*g*) therefore AB $\sqcup$ FD. *Which was to be dem.* But ſecondly, if AB $\sqcup$ FD, (*h*) then ſhall AB $\sqcup$ AB − FD (2 DB) (*k*) therefore AB $\sqcup$ DB, (*l*) wherefore AD $\sqcup$ DB. *Which was to be demonſtrated.* a 10. 1. b 28. 6. c 8. 2. d *conſtr.* & 4. 2. e 16. 10. f *conſtr.* g *cor.* 16. 10. h *cor.* 16. 10. k 12. 10. l 16. 10.

## PROP. XIX. Plate IV. *Fig.* 24.

*If there are two right-lines unequal* AB, GK, *and to the greater* AB, *a parallelogram* ADB *equal to the fourth part of a ſquare made upon the leſs* GK, *and deficient in figure by a ſquare be applied, and divides the ſaid* AB, *into parts* AD, DB, *incommenſurable in length; the greater line* AB *ſhall be in power more than the leſs* GK *by the ſquare of the right-line* FD *incommenſurable to the greater in length. And if the greater line* AB *be more in power than the leſs* GK *by the ſquare of a right-line* FD *incommenſurable to it in length, and if alſo upon tho greater* A B *be applied a parallelogram* A D B *equal to the fourth part of the ſquare of the leſs* G K, *and deficient in figure by a ſquare, then ſhall it divide the ſaid greater line* A B, *into parts incommenſurable in length* A D, D B.

Suppoſe all the ſame that was done and ſaid in the
a 17. 10. prec. prop. Therefore firſt, If AD ⌐⊔ D B, (*a*) then
b 13. 10. ſhall AB ⌐⊔ DB. (*b*) Wherefore AB ⌐⊔ 2 D B (A B −
c *cor.* 17. 10. FD) (*c*) therefore A B ⌐⊔ FD. *Whith was to be demonſtrated.*
d 13. 10. Secondly, If A B ⌐⊔ F D, then A B ⌐⊔ A B — F D
e 17. 10. ( 2 DB ) (*d*) wherefore AB ⌐⊔ DB, (*e*) and conſequently AD ⌐⊔ DB. *Which was to be demonſtrated.*

## PROP. XX. *Fig.* 25.

*A rectangle* BD *comprehended under right-lines* BC, CD, *rational and commenſurable in length according to one of the foreſaid ways, is rational.*

a 46. 1. Let A be given ρ, and (*a*) the ſquare BE deſcribed up-
b 1. 6. on BC. Becauſe DC : CE (BC) (*b*) : BD : BE, and DC
c *hyp.* (*c*) ⊔ BC, (*d*) therefore ſhall the rectangle B D be ⊔
d 10. 10. ſquare B E, wherefore ſeeing the ſquare B E (*e*) ⊔ Aq,
e *hyp.* & 9. *def.* 10. ſhall alſo (*f*) BD be ⊔ Aq, and ſo the rectangle ρν BD.
f 12. 10. *Which was to be demonſtrated.*

Note, *There are three kinds of rational-lines commenſurable one to another. For either of two rational-lines commenſurable in length one to the other, one is equal to the rational-line propounded, or either of them is equal to it, notwithſtanding both of them are commenſurable to it in length; or laſtly both of them are commenſurable to the rational-line given only in power. And theſe are the ways which the preſent Theorem ſpeaks of.*

In numbers, let there be BC, √ 8(2 √ 2) and CD √ 18(3 √ 2) then ſhall the rectangle BD = √ 144 = 12.

## PROP. XXI. Plate IV. *Fig.* 26.

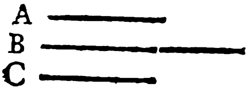
*If a rational rectangle* DB *be applied to a rational line* DC, *it makes the breadth thereof* CB *rational, and commenſurable in length to that line* DC *whereto* DB *is applied.*

Let G be propounded ῥ, and the ſquare DA deſcribed on BC, becauſe BD : DA : : (*a*) BC : CA; and BD, DA (*b*) are ῥs. (*c*) and ſo ⊔ (*d*) therefore BC ⊔ CA but CD (CA) is ῥ. (*e*) therefore BC is ῥ. *Which was to be demonſtrated.*
a 1. 6.
b *hyp.*
c *ſch.* 12. 10.
d 10. 10.
e *ſch.* 12. 10.

In numbers, let there be the rectangle DB, 12, and DC, √ 8. then ſhall CB, √ 18. but √ 18 = 3 √ 2. and √ 8 = 2 X √ 2.

### *Lemma.*

*To find out two rational right-lines commenſurable only in power.*



Let A be proponnded ῥ. (*a*) Take B ⊔ A, (*a*) and C ⊔ B, (*b*) it is clear that B and C are the lines required.
a 11. 10.
b *ſch.* 12. 10.

## PROP. XXII. *Fig.* 26.

*A rectangle* DB *comprehended under the rational right-lines* DC, CB *commenſurable in power only, is irrational: and the right-line* H, *which containeth that rectangle in power is irrational, and called a Medial-line.*

Let G be the propounded ῥ, and the ſquare DA deſcribed on DC, and let Hq = DB. Becauſe AC : CB (*a*) : : DA : DB, (*b*) and AC ⊔ CB, (*c*) ſhall be DA ⊔ DB (Hq.) (*d*) but Gq ⊔ DA; (*e*) therefore Hq ⊔ Gq (*f*) wherefore H is ῥ. *Which was to be dem.* and let it be called a Medial-line, becauſe AC : H : : H : CB.
a 1. 6.
b *hyp.*
c 10. 10.
d *hyp.* and 9 *def.* 10.
e 13. 10.
f 11. 10.

In numbers, let there be DC, 3. and CB, √ 6. then ſhall the rectangle be DB (Hq) √ 54. wherefore H is √ 54.

The note of a medial-line is μ, of a medial-rectangle μν, of both together μα.

*Schol.*

### *Schol.*

Every rectangle that can be contained under two rational right-lines commenſurable only in power, is medial, although it be contained under two right-lines irrational: and every medial-rectangle may be contained under two rational right-lines, commenſurable only in power; as for example, the $\sqrt{24}$ is μ, becauſe it is contained under $\sqrt{3}$, and $\sqrt{8}$, which are ῥ, ⌐ʒ although it may be contained under $v\sqrt{6}$, and $v\sqrt{96}$ irrationals, for $\sqrt{24} = v\sqrt{576} = v\sqrt{6} \times v\sqrt{96}$.

## PROP. XXIII. Plate IV. *Fig.* 27.

*If the rectangle* BD *made of a medial-line* A, *be applied to a rational-line* BC, *it makes the breadth* CD *rational, and incommenſurable in length to the line* BC, *whereunto the rectangle* BD *is applied.*

a *ſch.* 22. 10.
b 1. *ax.* 1,
c 14. 6.
d 22. 6.
e *hyp.*
f *ſch.* 12. 10.
g 10. 10.
h *ſch.* 12. 10.
k 1. 6.
l 10. 10.
m *ſch.* 12. 10.
n 13. 10.
o 1. 6.
p 10. 10.

Becauſe A μ, (*a*) therefore ſhall Aq be equal to ſome rectangle (EG) contained under EF and FG ῥ ⌐ʒ. (*b*) therefore BD=EG, (*c*) whence BC : EF :: FG : CD (*d*) therefore BCq : EFq : : FGq : CDq. But BCq and EFq (*e*) are ῥν. (*f*) and ſo ⌐⊓. (*g*) Therefore FGq ⌐⊓ CDq. Wherefore ſince FG is ῥ, (*h*) therefore CD ſhall be ῥ. Moreover, becauſe EF : FG (*k*) : : EFq : EG (BD;) for ſince EF ⌒⊓ FG, (*e*) ſhall EFq be ⌒⊓ BD. But EFq (*m*) ⌒⊓ CDq. (*n*) therefore the rectangle BD ⌒⊓ CDq. Whence ſince CDq : BD (*o*) : : CD : BC. (*p*) ſhall CD be ⌒⊓ BC, therefore, &c.

## PROP. XXIV. *Fig.* 28.

*A right-line* B *commenſurable to a medial-line* A, *is alſo a medial-line.*

a 11. 6.
b *hyp.*
c 23. 10.
d 1. 6.
e *hyp.*
f 10. 10.
g 12. *and* 13. 10.
h 22. 10.

Upon CD ῥ (*a*) make the rectangle CE=Aq; (*a*) and the rectangle CF=Bq. Becauſe Aq (CE) is μν, (*b*) and CD ῥ, (*c*) therefore ſhall the latitude DE be ῥ ⌐ʒ CD. But becauſe CE : CF (*d*) : : ED : DF and CE (*e*) ⌐⊓ CF, (*f*) therefore ED ⌐⊓ DF. (*g*) therefore DF is ῥ ⌐⊓ CD. (*h*) whence the rectangle CF (Bq) is μν, and ſo B is μ. *Which was to be demonſtrated.*

Obſ. *that the note* ⌐ʒ *for the moſt part ſignifies commenſurable in power only, as in this and the precedent demonſtrations,* &c.

### *Coroll.*

Hereby it is manifeſt that a ſpace commenſurable to a medial-ſpace, is alſo medial.

### *Lemma.* Plate IV. *Fig.* 21.

*To find out two medial right-lines* A, B, *commenſurable in length and alſo two* A C *commenſurable only in power.*

(*a*) Let A be any μ, (*b*) take B ⊤⊔ A, and (*c*) C ⊤⊓ A, (*d*) and 'tis evident the thing is is done.

a *lem.* 21. 10 *and* 13. 6.
b 2. *lem.* 10. 10.
c 3. *lem* 10. 10.
d *conſt. and* 24. 10.

### PROP. XXV. *Fig.* 29.

*A rectangle* DB *contained under* DC, CB *medial right-lines commenſurable in length is medial.*

Upon DC deſcribe the ſquare DA. Becauſe AC : (DC) CB (*a*) : : DA : DB, and DC ⊤⊔ CB ; (*b*) ſhall DA ⊤⊔ DB. (*c*) therefore DB is μν. *Which was to be demonſtrated.*

a 1. 6.
b 10. 10.
c 24. 10.

### PROP. XXVI. *Fig.* 30.

*A rectangle* AC *comprehended under medial right-lines* AB, BC *commenſurable only in power is either rational or medial.*

Upon the lines AB, BC, (*a*) deſcribe the ſquares AD, CE; and upon FG ῥ (*b*) make the rectangles FH = AD, (*b*) and IK=AC (*b*) and LM=CE.

The ſquares AD, CE, that is, the rectangles FH, LM, (*c*) are μα and ⊤⊔. therefore GH, KM, having the ſame proportion (*d*) are ῥ, (*e*) and ⊤⊔ (*f*) therefore GH×KM is ῥν. But becauſe AD, AC, CE, that is FH, IK, LM, (*g*) are ∺ ; (*h*) and ſo GH, HK, KM alſo ∺ ; (*k*) thence HKq = GH × KM. (*l*) therefore HK is ῥ, or ⊤⊔, or ⊤⊓ IH (GF ;) if ⊤⊔, (*m*) then the rectangle IK or AC is ῥν, but if ⊤⊓ (*n*) then AC is μν. *Which was to be dem.*

a 46. 1.
b *cor.* 16. 6.
c *hyp.* & 24. 10.
d 23. 10.
e 10. 10.
f 20. 10.
g *ſch.* 22. 6.
h 1. 6.
k 17. 6.
l 12. 10.
m 20. 10.
n 22. 10.

### *Lemma. Fig.* 31.

*If* A *and* E *are* ⊤⊓ *only, Then firſt,* ſhall Aq, Eq, Aq + Eq, Aq—Eq (*a*) ⊤⊔. *And ſecondly* Aq, Eq, Aq + Eq, Aq—Eq ⊤⊔ AE and 2 AE. For A : E (*b*) : : Aq. AE (*b*) : : AE : Eq. therefore ſeeing A (*c*) ⊤⊔ E, (*d*) ſhall Aq ⊤⊔ AE, (*e*) and 2 AE. Alſo Eq (*d*) ⊤⊔ AE, (*e*) and 2 AE. wherefore becauſe Aq + Eq ⊤⊔ Aq and Eq; and Aq—Eq ⊤⊔ Aq and Eq ; (*f*) therefore ſhall Aq + Eq, (*f*) and Aq — Eq be ⊤⊔ AE, and 2 AE. *Hence*

a *hyp. and* 16. 10.
b 1. 2.
c *hyp.*
d 10. 10.
e 14. 10.
f 14. 10.

*Hence also thirdly,* Aq, Eq, Aq + Eq, Aq − Eq, 2
g 14. 10. AE (*g*) ⊓ Aq + Eq + 2 AE; and Aq + Eq − 2
*and* 7. 10. AE. (*g*) and Aq + Eq + 2 AE ⊓ Aq + Eq − 2
h *cor.* 7. 2. AE. (*h*) (Q. A − E.)

## PROP. XXVII. Plate IV. *Fig.* 32.

*A medial rectangle* AB *exceedeth not a medial rectangle* AC *by a rational rectangle* DB.

a *cor.* 16. 6. Upon EF ρ, (*a*) make EG = AB, (*a*) and EH = AC.
b *hyp.* The rectangles AB, AC, *i. e.* EG, EH, (*b*) are μα; (*c*)
c 3. 10. therefore FG and FH are ρ́ ⊔ EF. Whence, if KG,
d 3. *ax.* 1. (*d*) *i. e.* DB be ρν, (*e*) then ſhall HG be ⊓ HK; (*f*)
e 1. 10. wherefore HG ⊔ FH. (*g*) and conſequently FGq ⊓
f 13. 10. FHq. But FH is ρ́: (*h*) therefore is FG ρ. but FG was
g *lem.* 26. 10. ρ́ before. *Which is contradictory.*
h *ſch.* 12. 10.

### *Schol.* *Fig.* 29, 33.

1. *A rational rectangle* A E *exceeds a rational rectangle* AD *by a rational rectangle* CE.

a *hyp.* For AE (*a*) ⊓ AD, (*b*) therefore AE ⊓ CE (*c*)
b *cor.* 16. 10. wherefore CE is ρν. *Which was to be demonſtrated.*

2. *A rational rectangle* AD *joined with a rational rectangle* CF *makes a rational rectangle* AF.

c *ſch.* 12. 10.
a *ſch.* 12. 10. For AD (*a*) ⊓ CF, (*b*) wherefore AF ⊓ AD and
b 16. 10. CF; (*c*) and ſo AF is ρν. *Which was to be demonſtrated.*
c *ſch.* 12. 10.

## PROP. XXVIII. *Fig.* 22.

*To find out medial-lines* (C *and* D) *which contain a rational rectangle* CD.

a *lem.* 21. 10. (*a*) Take A and B ρ́ ⊓, (*b*) make A : C :: C : B, (*c*)
b 13. 6. and A : B :: C : D I ſay the thing required is done.
c 12. 6. For AB (Cq) (*d*) is μν, (*d*) whence C is μ. But becauſe
d 22. 10. A : B :: C : D (*f*) therefore C ⊓ D; (*g*) and conſe-
e *conſtr.* quently D is μ. Moreover by permutation A : C ::
f 10. 10. B : D. *i. e.* C : B :: B : D; (*h*) therefore Bq = CD.
g 24. 10. But Bq is ρν; (*h*) therefore CD is ρν. *Which was to be*
h 17. 6. *done.*
h *ſch.* 12. 10.

In numbers, let A be √2; and B √6. therefore C is v√12. make √2 : √6 :: v√12 : D. or v4 : v√36 :: v√12 : D; then ſhall D be v√108; but v√12 × v√108 = v√1296 = √36 = 6. therefore CD is 6, likewiſe C : D :: 1 : √3; wherefore C ⊔ D.

PROP.

## PROP. XXIX. Plate IV. *Fig.* 34.

*To find out medial right-lines commenſurable in power only,* D *and* E, *containing a medial rectangle* DE.

(*a*) Take A, B, C ῥ ⊤, make A : D (*b*) : : D : B. (*c*) and B : C : : D : E I ſay the thing deſired is performed. For AB (*d*) = Dq ; and AB (*e*) is *μr*, therefore D is *μ* ; and B (*f*) ⊤ C, (*g*) whence D ⊤ E. (*h*) E is *μ*. Moreover B : C (*f*) : : D : E, and by permutation B : D : : C : E. *i. e.* D : A : : C : E ; (*l*) therefore DE = AC. But AC (*m*) is *μr* ; therefore DE is *μr*. *Which was to be done.*

In numbers, let A be 20, and B, $\sqrt{}$ 200, and C, $\sqrt{}$ 80. Therefore D is $\sqrt{}^{4}$ 80000 ; and E v $\sqrt{}$ 12800. Therefore DE = $\sqrt{}^{14}$ 1024000000 = $\sqrt{}$ 32000. and D : E : : $\sqrt{}$ 10 : 2. wherefore D ⊤ E.

a lem 21. 10.
b 13. 6.
c 12. 6.
d 17. 6.
e 22. 10.
f *conſtr.*
g 10. 10.
h 24. 10.
k *conſtr.*
*and cor* 4. 5.
l 16. 6.
m 22. 6.

### *Schol.*

*To find out two plane numbers, like or unlike,*

Take any four numbers proportional A : B : : C : D it is manifeſt that AB and CD are like plane numbers. And you may find out as many unlike plane numbers, as you pleaſe, by the help of *Schol*, 27. 8.

A, 6. C 12
B, 4. D, 8.
AB, 24. CD, 96.

A, 6. C, 5.
B, 4. D, 8.
AB, 24. CD, 40.

### *Lemma.* *Fig.* 35.

*To find out two ſquare numbers* (DEq *and* CDq) *ſo that the number compoſed of them* (CEq) *be ſquare alſo.*

Take AD, DB like plane numbers (of which let both be even, or both odd) *viz.* AD, 24. and DB 6. The total of theſe (AB) is 30 ; the difference (FD) 18. Half of which (CD) is 9. (*a*) Now the like plane numbers AD, DB, have one mean number proportional, namely DE ; therefore it is evident that every of thoſe numbers CE, CD, DE, are rational, and by conſequence CEq (*b*) (CDq + DEq ) is the ſquare number required.

a 18. 8.
b 47. 1.

Whereby it will be eaſy to find out two ſquare numbers, the exceſs of which is a ſquare or not a ſquare number, namely by the ſame conſtruction (*c*) ſhall CEq — CDq be = DEq.

c 3. *ax.* 1.

But if AD, DB be plane numbers unlike, the mean proportional line (DE) ſhall not be a rational number, and ſo neither ſhall the exceſs (DEq) of the ſquare numbers, CEq, CDq, be a ſquare number.

*Lemma* 2.

2. *To find out two ſuch ſquare numbers* B, C, *as the number compounded of them* D *is not ſquare, Alſo to divide a ſquare number* A *into two numbers* B, C, *not ſquares.*

A, 3. B, 9. C, 36. D, 45.

1. Take any ſquare number B, and let C be $=$ 4 B, and D $=$ B $+$ C. I ſay the thing is done.

For B is Q, by the conſtr. likewiſe becauſe B : C : : 1
a 24. 8, : 4 : : Q : Q; (*a*) therefore C alſo ſhall be a ſquare
b *cor.* 24. 8. number. But becauſe B $+$ C (D) : C : : 5 : 4 : : not
Q : Q. (*b*) therefore ſhall not D be a ſquare number. *Which was to be done.*

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be ſome ſquare number. Take D, E, F, plane numbers unlike, and let D be $=$ E $+$ F. make D : E : : A : B. and D : F : : A : C. I ſay the thing required is done.

For becauſe D : E $+$ F : : A : B $+$ C, and D $=$ E $+$
a 14. 5. F, (*a*) therefore ſhall A $=$ B $+$ C. Now ſuppoſe B to
b 21. *def.* 7. be ſquare, (*b*) then A and B, (*c*) and conſequently D and
c 26. 8. E are like plane numbers. *Which is contrary to the Hyp.*

The ſame abſurdity will follow if C be ſuppoſed a ſquare number, Therefore, *&c.*

## PROP. XXX. *Fig.* 36.

*To find out two rational right-lines* AB, AF, *commenſurable only in power, ſo that the greater* AB *ſhall be in power, more than the leſs* A F, *by the ſquare of a right-line* BF, *commenſurable to it ſelf in length.*

a 1. *lem.* 29. Let A B be ῥ. (*a*) Take the ſquare numbers C D,
10. CE, ſo that CD — CE (ED) be not Q. (*b*) and make
b 3. *lem.* 10. CD : ED : : ABq : AFq. In a circle deſcribed upon
10. the diameter AB (*c*) fit AF, and draw BF. Then I ſay
c 1. 4. AB, AF, are the lines required.

For ABq : AFq (*d*) : : CD : ED ; (*e*) therefore ABq ⊏⊐ AFq. But AB is ρ̇ ; (*f*) therefore AF is alſo ρ̇. But becauſe CD is Q: and ED not Q: (*g*) therefore ſhall AB be ⊏⊐ AF. Moreover by reaſon of the (*h*) right-angle AFB, is ABq (*k*) = AFq + BFq ; therefore ſeeing ABq : AFq : : CD : ED, by converſion of proportion ſhall ABq : BFq : : CD : CE : : Q: Q. (*l*) Therefore AB ⊏⊐ BF. *Which was to be done.* d *conſtr.* e 6. 10. f *ſch.* 12. 10. g 9. 10. h 31. 3. k 47. 1. l 9. 10.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9 : 5 : : 36 : (Q: 6.) AFq. then AFq ſhall be 20 ; and conſequently AF √ 20. Therefore BFq = 36 − 20 = 16. Wherefore BF is 4.

## PROP. XXXI. Plate IV. *Fig.* 36.

*To find out two rational lines* AB, AF *commenſurable only in power, ſo that the greater* AB *ſhall be in power more than the leſs* AF *by the ſquare of a right-line* BF *incommenſurable to it ſelf in length.*

Let AB be ρ̇. (*a*) Take the ſquare numbers CE, ED, ſo that CD = CE + ED be not Q, and in the reſt follow the conſtruction of the preced. *prop.* I ſay then the thing required is done. a 2 *lem.* 29. 10.

For, as above, AB, AF, are ρ̇ ⊓⊔ alſo ABq : BFq : : CD : ED. therefore ſince CD is not Q: AB, BF (*b*) ſhall be ⊏⊐. *Which was to be done.* b 9. 10.

In numbers, let there be AB, 5. CD, 45. CE = 36. ED = 9. Make 45 : 9 : : 25 (ABq.) : 5 (AFq) ; therefore AF = √5. conſequently BFq = 45 − 25 = 20. Wherefore BF = √ 20.

## PROP. XXXII. *Fig.* 37.

*To find out two medial-lines* C, D, *commenſurable only in power, comprehending a rational rectangle* CD, *ſo that the greater* C *be more in power than the leſſer* D *by the ſquare of a right-line commenſurable in length to the greater.*

(*a*) Take A and B ρ̇ ⊓⊔ ; ſo as √ Aq − Bq ⊏⊐ A (*b*) and make A : C : : C : B. (*c*) and A : B : : C : D. I ſay the thing is done. a 30. 10. b 13. 6. c 12. 6.

For becauſe A and (*d*) B are ρ̇ ⊓⊔ (*e*) therefore ſhall C (*f*) (√ AB) be μ. (*g*) and thence alſo C ⊓⊔ D (*h*) therefore D is likewiſe μ. Furthermore, whereas A : B (*d*) : : C : D ; and by permutation A : C : : B : D : : C : B ; and Bq is ρ̇ν ; therefore ſhall CD (*k*) (Bq) be ρ̇ν. Laſtly, becauſe √ Aq − Bq (*d*) ⊏⊐ A, (*l*) ſhall √ Cq − Dq be ⊏⊐ C, therefore, &c. But if √ Aq − Bq ⊏⊐ Aq, then ſhall √ Cq − Dq be ⊓⊔ C. d *conſtr.* e 22. 10. f 17. 6. g 10. 10. h 24. 10. k 17. 6. l 15. 10.

In numbers; let there be A 8, B √ 48 (√ : 64 — 16) therefore C = √ AB = v √ 3072. and D = v √ 1728, wherefore CD = v √ 5308416 = √ 2304.

### PROP. XXXIII. Plate IV. *Fig.* 38.

*To find out two medial-lines* D, E, *commenſurable in power only, comprehending a medial rectangle* DE, *ſo that the greater* D *ſhall be more in power than the leſs* E, *by the ſquare of a right-line commenſurable to the greater length.*

a 30. 10. (*a*) Take A and C ῤ ⊐, ſo that √ Aq — Cq ⊏⊐ A, (*b*)
b *lem.* 21. 10. take alſo B ⊐ A and C, and make A : D (*c*) : : D : B (*d*)
c 13. 6. : : C : E then D, and E are the lines ſought for.
d 12. 6. For becauſe A and C (*e*) are ῤ, (*e*) and B ⊐ A and C,
e *conſtr.* (*f*) therefore ſhall B be ῤ, and D (√ AB) (*g*) ſhall be μ.
f *ſch.* 12. 10. But becauſe A : D : : C : E, therefore by permutation A :
g 22. 10. C : : D : E. Wherefore ſeeing A ⊐ C, therefore D ſhall
h 10. 10. be ⊐ E; therefore E is μ. Furthermore, (*l*) becauſe
k 24. 10. D : B : : C : E. and BC is μν; alſo DE, equal to it, is μν.
l 22. 10. Laſtly, becauſe A : C : : D : E (*e*) ſeeing √ Aq — Cq ⊏⊐
m 16. 6. A; therefore √ Dq — Eq ⊏⊐ D; therefore, &c. But if √
n 15. 5. Aq — Cq ⊏⊐ A. then √ Dq — Eq ⊏⊐ Eq.

In numbers, let there be A 8, C √ 48, B √ 28. Then D v √ 3072. and E v √ 588. wherefore D : E : : 2 : √ 3. And DE = √ 1344.

### PROP. XXXIV. *Fig.* 39.

*To find ont two right-lines* AF, BF, *incommenſurable in power, whoſe ſquares added together make a rational figure, and the rectangle contained under them medial.*

a 31 10. (*a*) Let there be found AB, CD, ῤ ⊐; ſo that √ ABq
b 10. 1. — CDq ⊏⊐ AB; divide CD equally in G, (*c*) make the
c 28. 6. rectangle AEB = GCq. Upon AB, the diameter, draw
d 12. 6. a ſemicircle AFB, erect the perpendicular EF, and draw
e *cor.* 8. 6. AF, BF. Theſe are the lines required.
& 17. 6. For AE : BE (*d*) : : BA × AE : AB × BE. But BA ×
f 7. 5t AE (*e*) = AFq; and AB × BE = FBq. (*f*) Therefore AE :
g 19. 10. EB : : AFq : FBq; therefore ſince AE (*g*) ⊏⊐ EB, (*h*)
h 10. 10. AFq ſhall be ⊏⊐ FBq. Moreover ABq (*k*) AFq + FBq
k 31. 3. & 47. 1. (*l*) is ῤν. Laſtly EFq (*l*) = AEB (*l*) = CGq; (*m*) therefore
l *conſtr.* EF = CG. Therefore DE × AB = 2 EF × AB. But CD
m 1. ax. 1. × AB (*n*) is μν. (*o*) Therefore AB × ED, EF, (*p*) or AF ×
n 22. 10. FB is μν. *Which was to be demonſtrated.*
o 24. 10.
p *ſch.* 22. 6.

*The Explanation of the ſame by numbers.*

Let AB be 6, CD $\surd\,12$; then CG $=\surd\frac{12}{4}=\surd\,3$. But AE $=3+\surd\,6$. and EB $=3,-\surd\,6$, whence AF ſhall be $\surd^{4}\,18+\surd\,216$. and FB $\surd^{4}\,18-\surd\,216$. Alſo AFq + FBq is 36, and AF × FB $=\surd\,108$.

But AE is found in this manner. Becauſe BA (6.) : AF : : AF : AE. therefore 6 AE = AFq = AEq + 3 (EFq) therefore 6 AE − AEq = 3. Put $3+e=$ AE. then $18+6e-9.-6e-ee$, that is, $9-ee=3$. or $ee=6$. wherefore $e=\surd\,6$. and ſo AE $=3+\surd\,6$.

### P R O P. XXXV. Plate IV. *Fig.* 40.

*To find out two right-lines* AE, EB, *incommenſurable in power; whoſe ſquares added together make a medial figure, and the rectangle contained under them rational.*

(*a*) Take AB and CF μ ⊓̡, ſo that AB × CF be ῥν, and √ ABq — CFq ⊓̲ AB, and let the reſt be done as in the prec. *prop*. AE, EB are the lines required, a 32. 10.

For, as it is ſhewn there, AEq ⊓̲ EBq. alſo ABq (AEq + EBq) μν. and laſtly AB × CF (*b*) is ῥν. (*c*) therefore alſo AB × DE, that is, AE × EB, is ῥν, therefore, &c. b *conſt.* c *ſch.* 12. 10. d *ſch.* 22. 6.

### P R O P. XXXVI. *Fig.* 41.

*To find out two right-lines* BA, AC, *incommenſurable in power, whoſe ſquares added together make a medial figure, and the rectangle alſo contained under them medial, and incommenſurable to the figure compoſed of the ſquares.*

(*a*) Take BC and EF μ ⊓̡, ſo that BC × EF be μν. and √ BCq—EFq ⊓̲ BC, and ſo forward, as in the prec. BA, AC, ſhall be the lines ſought for. a 33. 10.

For (as above) BAq ⊓̲ ACq, alſo BAq + ACq is μν. end BA × AC is μν. Laſtly, BC (*b*) ⊓̲ EF, and (*c*) ſo BC ⊓̲ EG; likewiſe BC : EG (*d*) : : BCq. : BC × EG (BC × AD, or BA × AC) (*e*) therefore BCq (ABq + ACq) ⊓̲ BA × AC. therefore, &c. b *conſtr.* c 13. 10. d 1. 6. e 14. 10.

### *Schol. Fig.* 41.

*To find out two medial-lines incommenſurable both in length and power.*

a 36. 10. (*a*) Take BC μ, and let BA × AC be μν, and ⊔ BCq
b 13. 6. (BAq + ACq) (*b*) make BA : H : : H : AC. then I ſay
c 17. 6. BC and H are μ ⊔. For BC is μ. (*a*) and BA × AC (*c*)
d 14. 10. (Hq) is μν. wherefore H is alſo μ. (*d*) Likewiſe BA × AC
⊔ BCq ; therefore Hq ⊔ BCq. therefore, *&c.*

## *Here begins the ſenaries of lines irrational by compoſition.*

### P R O P. XXXVII. Plate IV. *Fig.* 42.

*If two rational-lines* AB, BC, *commenſurable only in power, are added together, the whole line* AC *is irrational, and is called a binomial-line, or of two names.*

a *hyp.* For becauſe AB (*a*) ⊔ BC, thence (*b*) ſhall ACq be
b *lem.* 26. 10. ⊔ ABq. But AB (*a*) is ῥ. (*c*) therefore AC is ῤ.
c 11. *def.* 10. *Which was to be demonſtrated.*

### P R O P. XXXVIII. *Fig.* 42.

*If two medial-lines* AB, BC, *commenſurable in power only, are compounded; and contain a rational rectangle, the whole line* AC *is irrational and called a firſt bimedial-line.*

a *hyp.* For becauſe AB (*a*) ⊔ BC, (*b*) ſhall ACq be ⊔ AB
b *lem.* 26. 10. × BC, ῥν. (*c*) therefore AC is ῤ. *Which was to be demo.*
c 11. *def.* 10.

### *Lemma. Fig.* 43.

*A rectangle* AC, *contained under a rational-line* AB *and an irrational-line* BC, *is irrational.*

a *hyp.* For if the rectangle AC be affirmed ῥν, (*a*) then be-
b 21. 10. cauſe AB is ῥ, (*b*) the breadth BC ſhall be alſo ῥ. *againſt the Hyp.*

### P R O P. XXXIX. *Fig.* 44.

*If two medial-lines* A B, B C, *commenſurable only in power, containing a medial rectangle, are compounded, the whole line* A C *ſhall be irrational, and is called a ſecond bimedial-line.*

a *cor.* 16. 6. Upon the propounded line DE ῥ (*a*) make the rectangle
b 47. 1. & 11. 6. DF = ACq ; (*b*) and DG = ABq + BCq.
c *hyp.* Becauſe ABq (*c*) ⊔ BCq, (*d*) therefore ABq + BCq, *i.e.*
d 16. 10. DG, ⊔ ABq : but ABq (*e*) is μν, (*e*) therefore DG is μν.
e 24. 10. But the rectangle ABC is taken μν, (*c*) and conſequently

2 ABC *(f)* (HF) is μν. *(g)* therefore EG and GF are ῥ. Alſo becauſe DG *(h)* ⊓⊔ HF; and DG : HF : : *(k)* EG : GF; *(l)* therefore EG ⊓⊔ GF. *(m)* therefore the whole EF is ῥ *(n)* wherefore the rectangle DF is ῥν. *(o)* therefore √ DF, *i. e.* AC, is ρ̊, *Which was to be demonſtrated.*

f 4. 2.
g 23. 10.
h *lem.* 26. 10.
k 1. 6.
l 10. 10.
m 37. 10.
n *lem.* 38. 10.
o 11. *def.* 10.

## PROP. XL. Plate IV. *Fig.* 42.

*If two right-lines* AB, BC, *commenſurable only in power, are added together, making that which is compoſed of their ſquares rational, and the rectangle contained under them medial, the whole right-line* AC *is irrational, and is called a Major-line.*

For whereas ABq + BCq *(a)* is ῥν, and *(b)* ⊓⊔ 2 ABC *(c)* μν; and ſo ACq *(d)* (ABq + BCq + 2 ABC) *(e)* ⊓⊔ ABq + BCq ῥν. *(f)* therefore ſhall AC be ρ̊. *Which was to be demonſtrated.*

a *hyp.*
b *ſch.* 12. 10.
c *hyp. and* 24. 10.
d 4. 2.
e 17. 10.
f 11. *def.* 10.

## PROP. XLI. *Fig.* 45.

*If two right-lines* AC, CB, *incommenſurable in power, are added together, having that which is made of their ſquares added together medial, and the rectangle contained under them rational, the whole right-line* AB *ſhall be irrational, and is called a line containing in power a rational and a medial rectangle.*

For 2 rectangles ACB *(a)* ῥν, *(b)* ⊓⊔ ACq + CBq *(c)* μν. *(d)* therefore 2 ACB *(d)* ⊓⊔ ABq. wherefore *(e)* AB is ρ̊. *Which was to be demonſtrated.*

a *hyp. and ſch.* 12. 10.
b *ſch.* 12. 10.
c *hyp.*
d 17. 10.
e 11. *def.* 10.

## PROP. XLII. *Fig.* 46.

*If two right-lines* GH, HK, *incommenſurable in power, are added together, having both that which is compoſed of their ſquares medial, and the rectangle contained under them medial, and incommenſurable to that which is compoſod of their ſquares, the whole right-line* GK *is irrational, and is called a line containing in power two medial figures.*

Upon the propounded line FB ῥ make the rectangles AF = GKq, and CF = GHq + HKq. Becauſe GHq + HKq (CF) *(a)* is μν, the breadth CB *(b)* ſhall be ῥ. Alſo becauſe 2 rectangles GHK *(c)* (AD) *(a)* is μν, therefore AC *(b)* ſhall be ῥ. Moreover becauſe the rectangle AD *(a)* ⊓⊔. CF, *(d)* and AD : CF : : AC : CB, *(e)* thence ſhall AC be ⊓⊔ CB. *(f)* wherefore A is *(g)* ρ̊. therefore the rectangle AF. *i. e.* GKq is ρ̊ν; *(h)* and conſequently GK is ρ̊. *Which was to be demonſtrated.*

a *hyp.*
b 23. 10.
c 4. 2
d 2. 6.
e 10. 10.
f 37. 10.
g *lem.* 38. 10.
h 11. *def.* 10.

## PROP. XLIII. *Fig.* 47.

*A line of two names, or binominal,* AB, *can at one point only* D *be divided into its names,* AD, DB.

If it be poſſible, let the binominal-line AB be divided at the point E, into other names AE, EB. It is manifeſt that the line AB is in both caſes divided unequally, ſince AD ⊏⊐ DB, and AE ⊏⊐ EB.

a 37. 10. Becauſe the rectangles ADB, AEB (*a*) are μα ; (*a*) and
b *ſch.* 27. each of ADq, DBq, AEq, EBq is ῥν, (*b*) and ſo ADq +
10 DBq (*b*) and AEq + EBq are alſo ῥα. (*b*) therefore ADq +
c *ſch.* 5. 2. DBq —: AEq + EBq (*c*) *i. e.* 2 AEB — 2 ADB is ῥν. (*d*)
d *ſch.* 12. therefore AEB — ADB is ῥν. therefore μν exceeds μν by
10. ῥν. (*e*) *Which is abſurd.*
e 27. 10.

## PROP. XLIV. *Fig.* 47.

*A firſt bimedial-line* AB *is in one point only* D *divide into its names* AD, DB.

Conceive AB to be divided into other names AE, EB,
a 38. 10. whereupon every one ADq, DBq, EBq, will be (*a*) μα.
b *ſch.* 27. and the rectangles ADB, AEB, and the doubles of them,
10. ῥα ; (*b*) therefore 2 AEB — 2 ADB. (*c*) *i. e.* ADq + DBq
c *ſch.* 5. 2. —: AEq + EBq is ῥν, (*d*) *Which is abſurd.*
d 27. 10.

## PROP. XLV. *Fig.* 48.

*A ſecond bimedial-line* AB, *is divided into its names* AC, CB, *only at one point* C.

Suppoſe there were other names AD, DB. Upon the propounded line EF ῥ make the rectangles EG = ABq, and EH = ACq + CBq, as alſo EK = ADq + DBq.
a 39. 10. Becauſe ACq, BCq (*a*) are μα ⊏⊐ ; (*b*) ACq + CBq
b 16. *and* (EH) ſhall be μν. (*c*) therefore the breadth FH is ῥ. (*a*)
24. 10. Moreover the rectangle ACB, (*d*) and ſo 2 ACB (*e*) (IG)
c 23. 10. is μν. (*c*) therefore HG is alſo ῥ. And ſince EH is (*f*)
d 24. 10. ⊏⊐ IG, (*g*) and EH : IG : : FH : HG. (*h*) therefore FH,
e 4. 2. HG ſhall be ⊏⊐ (*k*) therefore FG is a binomial, whoſe
f *lem.* 26. 10 names are FH, HG. By the ſame reaſon FG is binomi-
g 1. 6. al, and the names of it FK, KG : *contrary to the* 43 *of this*
h 10. 10. *Book.*
k 37. 10.

## PROP. XLVI. Plate IV. *Fig.* 47.

*A major-line* AB *is at one point only* D *divided into its names* AD, DB.

Imagine other names AE, EB, whereupon the rectangles ADB, AEB, (*a*) μα. (*a*) and as well ADq + DBq, as AEq + EBq are ῥα. (*b*) therefore ADq + DBq — : AEq + EBq, (*c*) *i. e.* 2 AEB — 2 ADB is ῥν. (*d*) *Which is impoſſible.*

a 40. 10.
b *ſch.* 27. 10.
c *ſch.* 5. 21.
d 17. 10.

## PROP. XLVII. *Fig.* 49.

*A line* AB *containing in power a rational, and a medial-figure is divided at one point only* D *into its names* AD, DB.

Conceive other names AE, EB, then both AEq + EBq, and ADq + DBq are μα. (*a*) and the rectangles AEB, ADB are ῥα. (*b*) therefore 2 AEB — 2 ADB, (*c*) *i. e.* ADq + DBq, AEq + EBq is ῥν (*d*) *Which is abſurd.*

a 41. 10.
b *ſch.* 27. 10.
c *ſch.* 5. 2.
d 27. 10.

## PROP. XLVIII. *Fig,* 48.

*A line* AB *containing in power two medial-rectangles, is at one point only* C *divided into its names* AC, CB.

If you would have AB to be divided into other names AD, DB, draw upon the line compounded EF ῥ the rectangles EG = ABq, and EH = ACq + CBq, and EH = ADq + DBq. then becauſe ACq + CBq, namely EH, (*a*) is μν, (*b*) the breadth FH ſhall be ῥ. Alſo becauſe 2 ACB, (*c*) that is, IG, is (*a*) μν, HG (*b*) ſhall be likewiſe ῥ. Therefore, whereas EH (*a*) ⊔ IG, and EH : IG (*d*) : : FH : HG, thence FH (*e*) ſhall be ⊔ HG, (*f*) therefore FG is a binomial, and the names of it FH, HG. In like manner FK, KG ſhall be the names of it, *againſt the* 43. *of this Book.*

a 42. 10.
b 23. 10.
c 4. 2.
d 1. 6.
e 10 10.
f 37. 10.

### *Second Definitions.*

A Rational-line being propounded, and the binomial divided into its names, the greater of whoſe names is more in power than the leſs by the ſquare of a right-line commenſurable to the greater in length; then

I. If the greater name be commenſurable in length to the rational-line propounded, the whole line is called a binomial-line.

II. But if the leſſer name be commenſurable in length to the rational-line propounded, the whole line is called a ſecond binomial.

III. If neither of the names be commenſurable in length to the rational-line propounded, it is called a third binomial.

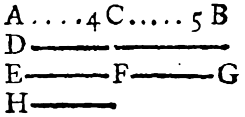
Furthermore, if the greater name be more in power than the leſs, by the ſquare of a right-line incommenſurable to the greater in length, then

IV. If the greater name be commenſurable to the propounded rational-line in length, it is called a fourth binomial.

V. If the leſſer name be ſo, a fifth.

VI. If neither, a ſixth.

## PROP. XLIX.



*To find out a firſt binomial-line,* EG.

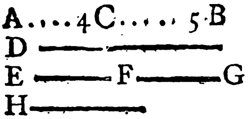
a *ſch.* 29. 10.
b 2 *lem.* 10. 10.
c 3 *lem.* 10. 10.
d *conſtr.*
e 6. *def.* 10.
f 6. 10.
g *ſch.* 12. 10.
h 9. 10.
k 9. 10.
l 1 *def.*48.10.

(*a*) Take AB, AC, ſquare numbers, whoſe exceſs CB is not Q. let D be propounded ῥ. (*b*) Take EF ⊓⊔ D, and (*c*) make AB : CB : : EFq : FGq. then EG ſhall be (*a*) firſt binomial.

For EF (*d*) ⊓⊔ D. (*e*) therefore EF is ῥ. (*f*) alſo EFq ⊓⊔ FGq. (*g*) therefore FG is alſo ῥ. likewiſe (*d*) becauſe EFq : FGq : : AB : CB : : Q : not Q (*b*) therefore EF ⊓⊔ FG. Laſtly, becauſe by converſion of proportion, EFq : EFq — FGq : : AB : AC : : Q : Q thence EF (*k*) ſhall be ⊓⊔ $\surd$ EFq — FGq. (*l*) therefore EG is a firſt binomial. *Which was to be done.*

In nnmbers thus ; let there be D 8. EF 6. AB 9. CB 5. wherefore becauſe 9 : 5 : : 36 : 20. therefore FG is $\surd$ 20. and conſequently EG is = 6 + $\surd$ 20.

## PROP. L.



*To find out a ſecond binomial-line,* EG.

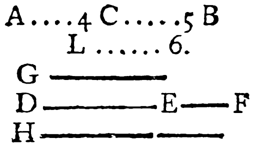
Take AB and AC ſquare numbers, the exceſs of which is CB not Q. Let the line D be propounded ῥ. take FG ⊓⊔ D, and make CB : AB : : FGq : EFq. then EG will be the line deſired.

*Prove it as the prec.*

For FG⊤⊔D. wherefore FG is ṕ. Alſo EFq ᒣ⊔ FGq therefore EF is ṕ. Likewiſe becauſe FGq : EFq : : CB : AB : : not Q : Q, thence FG is ᒣ⊔ EF. Laſtly, ſeeing CB : AB : : FGq : EFq, and inverſely AB : CB : : EFq : FGq. therefore as in the foregoing Prop. EF ⊤⊔ √ EFq — FGq. (*a*) whereby EG is a ſecond binomial. *Which was to be done.* a 2 *def.* 48. 10.

In numbers ; let there be D 8, FG 10, AB 9, CB 5 ; then EF is √ 180, wherefore EG is 10 + √ 180.

## PROP. LI.



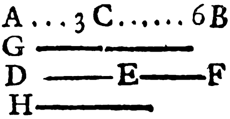
*To find out a third binomial line,* DF.

(*a*) Take AB, AC, ſquare numbers, the exceſs of which CB is not Q, and let L be a number not Q next greater than CB, *viz.* by a unit or two. Let G be the line propounded ṕ (*b*) Make L : AB : : Gq : DEq, (*b*) and AB : CB : : DEq : EFq, then DF ſhall be a third binomial. a *ſch.* 19. 10. b 3 *lem.* 10 10

For becauſe DEq (*c*) ⊤⊔ Gq, (*d*) DE is ṕ. alſo Gq : DEq : : L : AB : : not Q : Q. (*e*) therefore G ᒣ⊔ DE. Likewiſe ſince DEq (*e*) ⊤⊔ EFq, (*d*) alſo EF is ṕ. Moreover becauſe DEq : EFq : : AB : CB : : Q : not Q, (*f*) is DE ᒣ⊔ EF. and ſince by conſtr. and equality Gq : EFq : : L : CB : : not Q : Q. (for (*g*) L and CB are not like plane numbers) (*h*) therefore ſhall G be alſo ᒣ⊔ EF. Laſtly, as in the prec. prop. √ DEq—EFq ⊤⊔ DE. (*k*) therefore DF is a third binomial. *Which was to be done.* c *conſtr.* 6. 10. d *ſch.* 12. 10. e 6. 10. f 9. 10. g *ſch.* 27. 8. h 9. 10. k 3 *def.* 48. 10.

In numbers ; let there be AB, 9. CB, 5. L, 6. G, 8. then ſhall be DE √ 96, and EF $\sqrt{\frac{480}{9}}$. wherefore DF $= \sqrt{96} + \sqrt{\frac{480}{9}}$.

## PROP. LII.



*To find out a fourth binomial-line* DF.

(*a*) Take any ſquare number AB, and divide it into AC, CB not ſquares. Let G be the line propounded ṕ. (*b*) take DE ⊤⊔ G, (*c*) and make AB : CB : : DEq : EFq, then DF ſhall be a fourth binomial. a *ſch.* 29. 10. b 2. *lem.* 10 10. c 3 *lem.* 10. 10.

For, as in the 49 of this Book, DF may be ſhewn to be a binomial, and alſo becauſe by conſtr. and converſion of proportion DEq : DEq — EFq : : AB : AC : : Q: not Q. (*d*) ſhall DE be ⊔ √ DEq. — EFq (*e*) therefore DF is a fourth binomial.

d 9. 10.
e 4. *def.* 48. 10.

In numbers, let G be 8. DE, 6. then EF ſhall be $\sqrt{24}$. therefore DF is $6 + \sqrt{24}$.

## PROP. LIII.

A ... 3 C ...... 6 F.
G ———
D ——— E ——— F
H ———

*To find out a fifth binomial-line,* DF.

Take any ſquare number AB whoſe ſegments AC, CB are not Q. Let G be the line propounded ρ̊ take EF ⊔ G. and make CB : AB : : EFq : DEq. then ſhall DF be a fifth binomial.

For DF ſhall be a binomial as in the 50. of this Book, and becauſe by conſtruction, and inverſion, DEq : EFq : : AB ; CB, and ſo by converſion of proportion, DEq : DEq — EFq : : AB : AC : : Q : not Q (*a*) therefore ſhall DE be ⊔ √ DEq — EFq. (*b*) therefore DF is a fifth binomial. *Which was to be done.*

a 9. 10.
b 5. *def.* 48. 10.

## PROP. LIV.

A ..... 5 C ........ 7 B
L ......... 9
G ———————
D ——— E ——— F
H ———

*To find out a ſixth binomial-line.*

Take AC, CB, prime numbers, ſo that AC + CB (AB) be not Q. take alſo any number ſquare L. Let G be the line propounded ρ̊. (*a*) and make L : AB : : Gq : DEq, and AB : CB : : DEq : EFq. then DF ſhall be a ſixth binomial.

a 3. *lem.* 10. 10.

For DF may be demonſtrated binomial as in the 51. of this Book, and alſo by reaſon that DE and E ⊔ FG. Laſtly likewiſe becauſe by conſtr. and converſion of proportion DEq : DEq — EFq : : AB : AC : : not Q : Q. (For AB is prime to AC, (*b*) and ſo unlike to it) (*c*) therefore DE ⊔ √ DEq — EFq, (*d*) therefore DF is a ſixth binomial. *Which was required.*

b *ſch.* 27. 8.
c 9. 10.
d 6 *def.* 48. 10.

In numbers, let there be G 6. DE $\sqrt{48}$. then EF ſhall be $\sqrt{28}$. wherefore DF is $\sqrt{48} + \sqrt{28}$.

*Lemma.*

### *Lemma.* Plate V. *Fig.* 50.

*Let* AD *be a rectangle, and the ſide thereof* AC *divided*
*unequally in* E; *alſo let the leſſer portion* EC *be equal-*
*ly divided in* F. *Upon the line* AE (a) *make the rectangle* a 28.6.
AGE = EFq, *and from the points* G, E, F, (b) *draw* GH, b 31, 1.
EI, FK, *parallel to* AB. (c) *Let the ſquare* LM *be made* c 14. 2.
*equal to the rectangle* AH, *and upon* OMP *produced the*
*ſquare* MN = GI, *and let the right-lines* LOS, LQT,
NRS, NPT, *be produced.*

I ſay 1. MS, MT, are rectangles. For by reaſon of
the right-angles of the ſquares OMQ, RMP, (*a*) ſhall a *ſch.* 15.1
QMR be a right-line. (*b*) Therefore RMO, QMP, are b 13. 1.
right-angles, wherefore the parallelograms MS, MT, are
rectangles.

2. *Hence it is plain that* LS (*c*) = LT, *and conſequent-* c 2. 4. 1.
*ly that* LN *is a ſquare.*

3. *The rectangles* SM, MT, EK, FD, *are equal.* For
becauſe the rectangle AGE (*d*) = EFq. (*e*) thence ſhall d *hyp.*
AG : EF : : EF : GE, (*f*) and ſo AH : EK : : EK : GI, e 17. 6.
that is by conſtr. LM : EK : : EK : MN. (*g*) but LM : f 1. 6.
SM : : SM : MN; therefore EK (*h*) = SM (*k*) = FD g *ſch.*22.6.
(*l*) = MT. h 9. 5.

4. *Hence* LN (*m*) = AD. k 36. 1

5. *Becauſe* EC *is equally divided in* F, (n) *it is plain that* l 43. 1.
EF, FC, EC *are* ⊓⊔. m 2 *ax.* 1.

6. If AE ⊓̸⊔ EC, and AE ⊓⊔ √ AEq — ECq, (*o*) n 16. 10.
then ſhall AG, GE, AE, be ⊓⊔. alſo becauſe AG : GE o 18. *and*
: : AH : GI, (*p*) therefore ſhall AH, GI, *i. e.* LM, MN, 16. 10.
be ⊓⊔. Likewiſe p 10. 10.

7. OM ⸢⊓⊔ MP. For by the Hyp. AE ⸢⊓⊔ EC (*q*) q 14. 10.
therefore EC ⸢⊓⊔ GE (*q*) wherefore EF ⸢⊓⊔ GE. but EF r 10. 10.
: GE : : EK : GI (*r*) therefore EK ⸢⊓⊔ GI that is, SM ⸢⊓⊔
MN. But SM : MN : : OM : MP. (*r*) therefore OM ⸢⊓⊔
MP.

8. If AE be ſuppoſed ⸢⊓⊔ √ AEq — ECq, it is appa- ſ 19. *and*
rent that AG, GE, AE, are ⸢⊓⊔ whence (*ſ*) LM ⸢⊓⊔ MN. 17. 10.
for AG : GE : : AH : GI : : LM : MN.

*Theſe being well conſidered, we ſhall eaſily diſpatch the ſix following Propoſitions.*

## PROP. LV. Plate IV. *Fig.* 50.

*If a ſpace* AD *be contained under a rational-line* AB, *and a firſt binomial-line* AC (AE + EC) *the right-line* OP *which containeth that ſpace in power is irrational, and called a binomial-line.*

All that being ſuppoſed which is deſcribed and demonſtrated in the foregoing *Lemma*, it is manifeſt that the right-line OP containeth in power the ſpace AD. (*a*)
a *hyp. and* Likewiſe AG, GE, AE, are ⊓⊔. therefore ſeeing AE,
*lem.* 54. 10. (*b*) is ῥ ⊓⊔ AB, (*c*) ſhall alſo AG and GE be ῥ ⊓⊔ AB.
b *hyp.* (*d*) Therefore the rectangles AH, GI, that is, the ſquares
c *ſch.* 12. 10. LM, MN, are ῥα. therefore OM, MP are ῥ (*e*) ⊓⊔̦, (*f*)
d 20. 10. and conſeqently OP is a binomial. *Which was to be*
e *lem.* 54. 10. *demonſtrated.*

f 37. 10. In numbers, let there be AB5. AC4 + √ 12. wherefore the rectangle AD = 20 + √ 300 = to the ſquare LN. Therefore OP is √ 15 + √ 5. namely a ſixth binomial.

## PROP. LVI. *Fig.* 50.

*If a ſpace* AD *be comprehended under a rational-line* AB, *and a ſecond binomial* AC (AE + EC) *the right-line* OP, *which containeth that ſpace* AD *in power, is irrational, and called a firſt medial-line.*

The foreſaid *Lemma* of the 54 of this Book being again ſuppoſed, then ſhall OP be = √ AD. (*a*) alſo
a *hyp. and* AE, AG, GE, are ⊓⊔. therefore ſince AE (*b*) is ῥ ⊓⊔
*lem.* 54. 10. AB, likewiſe AG, GE (*c*) ſhall be ῥ ⊓⊔ AB, therefore
b *hyp.* the rectangles AH, GI, *i. e.* OMq, MPq. (*d*) are μα, (*e*)
c *ſch.* 12. 10. Moreover OM ⊓⊔̦ MP. Laſtly, EF ⊓⊔ EC, and EC
d 22. 10. (*f*) ⊓⊔ AB. (*g*) Wherefore EK, *i. e.* SM, or OMP, is
e *lem.* 54. 10. ῥν, (*h*) Conſequently OP is a firſt bimedial. *Which was*
f *hyp.* 12. *to be dem.*
10. In numbers, let there be AB 5, and AC √ 48 : + 6,
g 20. 10. then the rectangle AD = √ : 1200 + 30 = OPq. there-
h 38. 10. fore OP is v √ 675 + v √ 75, *viz.* a firſt bimedial.

## PROP. LVII. *Fig.* 50.

*If a ſpace* AD *be contained under a rational-line* AB, *and a third binomial-line* AC (AE + EC) *the right-line* OP *which containeth in power the ſpace* AD, *is irrational, and called a ſecond bimedial-line.*

As above, OPq = AD. Alſo the rectangles AH, GL, that is, OMq, MPq are ·· *a* (*a*) Likewiſe EK, or OMP is μν. (*b*) Therefore OP is a ſecond bimedial.
a *hyp. and* 22. 10.
b 39. 10.

In numbers, let there be AB 5. AC √ 32 + √ 24. wherefore AD is √ 800 + √ 600 = OPq. and ſo OP is *v* √ 450 + *v* √ 50, that is, a ſecond bimedial.

## PROP. LVIII. Plate IV. *Fig.* 50.

*If a ſpace* AD *be comprehended under a rational-line* AB, *and a fourth binomial* AC (AE + EC) *the right-line* OP *containing the ſpace* AD *in power, is that irrational-line which is called a Major-line.*

For again, OMq (*a*) ⊔ MPq; and the rectangle AI, *i. e.* OMq + MPq (*b*) is ῥν. (*c*) alſo EK or OMP is μν. (*d*) therefore OP (√ AD) is a *Major* line. *Which was to be demonſtrated.*
a *lem.* 54. 10.
b *hyp. and* 20. 10.
c *hyp. and* 22. 10.
d 40. 10.

In numbers, let there be AB 5. and AC 4 + √ 8. then the rectangle AD is 20 + √ 200. wherefore OP is √ : 20 + √ 200.

## PROP. LIX.

*If a ſpace* AD *be contained under a rational-line* AB, *and a fifth binomial* AC, *the right-line* OP *which containeth the ſpace* AD *in power, is that irrational-line, which is a line containing a rational and a medial rectangle in power.*

Again OMP ⊔ MPq. and the rectangle AI or OMq + MPq is μν. (*a*) Likewiſe the rectangle EK or OMP is ῥν. (*b*) therefore OP (√ AD) contains in power ῥν and μ'. *Which was to be demonſtrated.*
a *as in the prec.*
b 41. 10.

In numbers, let there be AB 5, and AC 2 + √ 8; then the rectangle AD = 10 + √ 200 = OPq. Wherefore OP is $\sqrt{\overline{10 + \sqrt{200}}}$.

## PROP. LX.

*If a ſpace* AD *be contained under a rational-line* AB *and a ſixth binomial* AC (AE + EC) *the line* OP *containing the ſpace* AD *in power is irrational, which containeth in power two medial-rectangles.*

As often before, OMq ⊔ MPq and OMq + MPq is μν. and alſo the rectangle (EK) OMP is μν. (*a*) therefore OP = √ AD contains in power 2 μα. *Which was to be demonſtrated.*
a 42. 10.

In numbers, let there be AB 5, AC √ 12 + √ 8, therefore the rectangle AD or OPq is √ 300 + √ 200 and ſo OP is √ : √ 300 + √ 200,

*Lemma*

### *Lemma.* Plate IV. *Fig.* 51.

*Let a right-line* AB *be unequally divided in* C, *and let* AC *be the greater ſegment, and upon ſome line* DE *apply the rectangles* DF = ABq, *and* DH = ACq, *and* IK = CBq, *and let* LG, *be divided equally in* M, *and alſo* MN *drawn parallel to* GF.

a 4. 2. *and* I ſay, 1. The rectangle ACB is = LN or MF. (*a*)
3. *ax.* 1. For 2 ACB = LF.
b 7. 2. 2. DL ⊏ LG. for DK (ACq + CBq) (*b*) ⊏ LF (2
c 1. 6. ACB) therefore ſince DK, LF are of equal altitude, (*c*)
DL ſhall be ⊏ LG.
d 16. 10. 3. If AC ⊔̡ CB, (*d*) then ſhall the rectangle DK be
⊔ ACq and CBq.
e *lem.* 26. 4. *Alſo* DL ⊔̸ LG. For ACq + CBq (*e*) ⊔̸ 2 ACB,
10. *i. e.* DK ⊔̸ LF, but DK : LF (*c*) : : DL : LG, (*f*)
f 10. 10. therefore DL ⊔̸ LG.
5. *Moreover* DL ⊔ √ DLq − LGq. For ACq : ACB
g 1. 6. (*g*) : : ACB : CBq. that is DH : LN : : IK. (*c*) LN :
h 17. 6. wherefore DL : LM : : LM : IL ; (*h*) therefore DI × IL
k *hyp.* = LMq. therefore ſeeing ACq (*k*) ⊔ CBq, that is, DH ⊔
l 10. 10. IK, and (*l*) ſo DI ⊔ IL, (*m*) ſhall DL be ⊔ √ DLq −
m 18. 10. LGq. *Whith was to be demonſtrated.*
n 19. 10. 6. *But if* ACq *be put* ⊔̸ CBq, (*n*) *then ſhall* DL be ⊔̸
√ DLq − LGq.

*This Lemma is preparatory to the ſix following Propoſitions.*

## P R O P. LXI.

*The ſquare of a binomial-line* (AC + CB) *applied unto a rational-line* DE, *makes the latitude* DG *a firſt binomial-line.*

a *hyp.* Thoſe things being ſuppoſed, which are deſcribed
b *lem.* 60. and demonſtrated in the preceding *Lemma*, becauſe AC,
10. CB (*a*) are ρ̊ ⊔̡, (*b*) the rectangle DK ſhall be ⊔
c *ſch.* 12. 10. ACq ; (*c*) and ſo DK is ρ̊ν ; (*d*) therefore DL ⊔ DE ρ̊.
d 21. 10. But the rectangle ACB, and ſo 2 ACB (LF) (*e*) *is* μν.
e 22. *and* (*f*) therefore the latitude LG is ρ̊ ⊔̸ DE, (*g*) therefore
24. 10. alſo DL ⊔̸ LG alſo DL ⊔ √ DLq − LGq. From
f 23. 10. whence (*k*) it follows that DG is a firſt binomial.
g 13. 10. *Which was to be demonſtrated.*
h *lem.* 60.
10.
k 1 *def.* 48.
10.

## PROP. LXII.

*The ſquare of a firſt bimedial-line* (AC+CB) *being applied to a rational-line* DE, *makes the latitude* DG *a ſecond binomial-line.*

The aforeſaid *Lemma* being again ſuppoſed; The rectangle DK ˥⊔ ACq; (*a*) therefore DK is μν; (*b*) therefore the latitude DL is ρ́ ʿ⊔ DE. But becauſe the rectangle ACB, and ſo LF (2 ACB) (*c*) is ρ́ν, (*d*) ſhall LG be ρ́ ˥⊔ DE; (*e*) therefore DL, LG are ˥⊔; (*f*) alſo DL ˥⊔ √ DLq — LGq; (*g*) from whence it is clear that DG is a ſecond binomial. *Which was to be demonſtrated,*

a 24. 10.
b 23. 10.
c *hyp. and ſch.* 22. 10.
d 21. 10.
e 13. 10.
f *lem.* 60. 10.
g 2. *def.* 48. 10.

## PROP. LXIII.

*The ſquare of a ſecond bimedial-line* (AC+CB) *applied to a rational-line* DE *makes the breadth* LG *a third binomial-line.*

As in the *prec.* DL is ρ́ ʿ⊔ DE. Furthermore becauſe the rectangle ACB, and ſo LF (2 ACB) (*a*) is μν; (*b*) therefore ſhall LG be ρ́ ʿ⊔ DE. (*c*) Moreover DL ʿ⊔ LG. and alſo DL ʿ⊔ √ DLq — LGq; (*d*) therefore DG is a third binomial, *Which was to be demonſtrated.*

a *hyp. and* 24. 10.
b 23. 10.
c *lem.* 60. 10.
d 3. *def.* 48. 10.

## PROP. LXIV.

*The ſquare of a Major-line* (AC+CB) *applied to a rational-line* DE, *makes the breadth* DG *a fourth binomial-line.*

Again ACq + CBq. *i. e.* DK (*a*) is ρ́ν, (*b*) therefore DL is ρ́ ˥⊔ DE, alſo ACB, and ſo LF (2 ACB) (*c*) is μν; (*d*) therefore LG is ρ́ ʿ⊔ DE, (*e*) and conſequently DL ʿ⊔ LG. Laſtly, becauſe AC ʿ⊔ BC, (*f*) ſhall DL be ʿ⊔ DLq — LGq, (*g*) whence DG is a fourth binomial. *Which was to be demonſtrated.*

a *hyp. and ſch.* 12. 10.
b 21. 10.
c *hyp. and* 24. 10.
d 23. 10.
e 13. 10.
f *lem.* 60. 10.
g 4. *def.* 48. 10.

## PROP. LXV.

*The ſquare of a line containing in power a rational and a medial rectangle* (AC + CB) *applied to a rational-line* DE *makes the latitude* DG *a fifth binomial.*

Again, DK is μν; (*a*) therefore DL is ρ́ ʿ⊔ DE; alſo LF is ρ́ν; (*b*) therefore LG is ρ́ ʿ⊔ DE; (*c*) therefore DL ʿ⊔ LG; (*d*) likewiſe DL ʿ⊔ √ DLq — LGq; (*e*) and ſo by conſequence DG is a fifth binomial. *Which was to be demonſtrated.*

a 23. 10.
b 21. 10.
c 13. 10.
d *lem.* 60. 10.
e 5. *def.* 48. 10.

PROP.

## PROP. LXVI.

*The ſquare of a line containing in power two medial rectangles* ( AB + CB ) *applied to a rational line* DE, *makes the latitude* DG *a ſixth binomial-line.*

As before, DL and LG are ρ '⊓⊔ DE. But becauſe
a *hyp.* ACq + CBq (DK) (*a*) '⊓⊔ ACB, (*b*) and ſo DK '⊓⊔ LF
b 14. 10. (2 ACB) and alſo DK : LF (*c*) : : DL : LG. (*d*) therefore
c 1. 6. ſhall DL be '⊓⊔ LG. (*e*) Laſtly DL '⊓⊔ √ DLq — LGq ;
d 10. 10. (*f*) by which it appears that DG is a ſixth binomial.
e *lem.* 60. 10.
f 6. *def.* 48. 10.

### *Lemma.*

*Let* AB, DE *be* ⊓⊔. *and make* AB : DE : : AC : DF.

I ſay 1. AC ⊓⊔ DF, as appears by 10. 10. alſo CB ⊓⊔
a 19. 5. FE ; (*a*) becauſe AB : DE : : CB : FE.

2. AC : CB : : DF : FE. For AC : DF : : AB : DE : : CB : FE ; therefore, by permutation, AC : CB : : DF : FE.

b 1. 6. 3. *The Rectangle* ACB ⊓⊔ DFE. For ACq : ACB (*b*)
c *before* : : AC : CB (*c*) : : DF : EF : : DFq : DFE, wherefore by permutation ACq : DFq : : ACB : DFE, therefore ſince
d 10. 10. ACq ⊓⊔ DFq, (*d*) ſhall ACB be ⊓⊔ DFE.

4. ACq + CBq ⊓⊔ DFq + FEq, For becauſe ACq
e 22. 6. : CBq. (*e*) : : DFq : FEq ; therefore by compounding ACq + CBq : CBq : : DFq + FEq : FEq, therefore ſince
f 10. 10. CBq ⊓⊔ FEq, (*f*) ſhall alſo ACq + CBq be ⊓⊔ DFq,
g 10. 10. + FEq.

5. *Hence,* If AC ⊓ᒍ or '⊓ᒍ CB, (*g*) then likewiſe ſhall DE be ⊓ᒍ or '⊓ᒍ EF.

## PROP. LXVII.

*A line* DE, *commenſurable in length to a binomial line* (AC + CB) *is it ſelf a*
a *lem.* 66. 10. *binomial-line, and of the ſame order.*

Make AB : DE : : AC : DF ; (*a*) then are AC, DF
b *hyp.* ⊓⊔ (*a*) and CB, FE ⊓⊔ ; whence ſince AC and CB, (*b*)
c *lem.* 66. 10. *and* ſch. 12. 10. are ρ ⊓ᒍ, (*c* thence, DF, FE ρ ⊓ᒍ ; therefore DF is a binomial. But becauſe AC : CB (*a*) : : DE : FE. If AC ⊓⊔ or '⊓⊔ √ ACq — BCq, (*d*) then in like manner
d 15. 10.

DF ⊔ or ʿ⊔ √ DFq − FEq ; alſo if AC ⊔ or ʿ⊔
ῥ propounded, (*e*) then ſhall DF be ⊔ or ʿ⊔ ῥ propound- e 12. 10.
ed. But if CB ⊔ or ʿ⊔ ῥ. likewiſe FE ʿ⊔ or ⊔ ῥ. *and* 14. 10.
If both AC, CB, ʿ⊔ ῥ, (*f*) then alſo both DF, FE, f *by def.* 48
ʿ⊔ ῥ. (*g*) That is, whatſoever binomial AB is, DE ſhall 10.
be of the ſame order. *Which was to be demonſtrated.* g 14. 10.

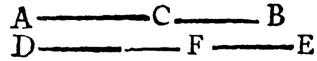
## PROP. LXVIII.

*A line* DE *commenſurable in length to a bimedial-line* (AC + CB) *is alſo a bimedial-line, and of the ſame order.*

Make AB : DE : : AC : DF ; (*b*) therefore AC ⊔ a 12. 6.
DF ; and CB ⊔ EF ; therefore ſeeing AC and CB (*c*) are b *lem.* 66.
μ, (*d*) alſo DF and FE ſhall be υ ; and becauſe AC (*e*) ⊓ 10.
CB, (*e*) therefore FD ⊓ FE. (*f*) therefore DE is 2ν. c *hyp.*
Wherefore if the rectangle ACB is ῥ ; becauſe DFE d 24. 10.
(*b*) ⊔ ACB, (*g*) likewiſe DFE is ῥν ; and if that be μν, e 10 10.
(*h*) this ſhall be μν too. (*k*) That is, whether AB be 1 f 38. 10.
bimed, or 2 bimed, DF ſhall be of the ſame order. *Which* g *ſch.* 12.
*was to be dem.* 10.

## PROP. LXIX.

h 24. 10.
k 38. *or* 39
10.



*A line* DE *commenſurable to a Major-line* (AC + CB) *is it ſelf a Major-line.*

Make AB : DE : : AC : DF. Becauſe AC (*a*) ʿ⊓ a *hyp.*
CB, (*b*) thence DF ʿ⊓ FE ; alſo ACq + CBq (*a*) is ῥν ; b *lem.* 66.
and ſo becauſe DFq + FEq (*b*) ⊔ ACq + CBq ; (*c*) 10.
alſo DFq + FEq is ῥν ; laſtly, the rectangle ACB (*a*) is c *ſch.* 12. 10.
μν ; (*d*) therefore the rectangle DFE is μν (becauſe d 24. 10.
DFE is (*b*) ⊔ A C B) (*e*) wherefore D E is a *Major-* e 40. 10.
*line. Which was to be demonſtrated.*

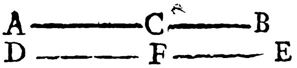
## PROP. LXX.

*A line* DE *commenſuarble to a line containing in power a rational and a medial-rectangle* (AC + CB) *is a line containing in power a rational and a medial-rectangle.*

Again make AB : DE : : AC : DF. Becauſe AC (*a*) a *hyp.*
⊓ CB, (*b*) alſo DF ʿ⊔ FE ; likewiſe becauſe ACq + b *lem.* 66.
CBq (*a*) is μν (*c*) therefore DFq + FEq ſhall be μν ; 10.
laſtly becauſe the rectangle ACB (*a*) is ῥν, (*d*) alſo DFE c 24. 10.
is ῥν. Therefore DE contains in power ῥν and μν. *Which* d *ſch.* 12. 10
*was to be dem.*

PROP.

## PROP. LXXI.



*A line* DE *commenſurable to a line containing two medial-rectangles in power* (AC+CB) *is alſo a line containing in power two medial-rectangles.*

a *hyp.* Divide DE, as in the prec. Becauſe ACq (*a*) ⊔ CBq,
b *lem.* 66. (*b*) thence ſhall DFq be ⊔ FEq ; alſo becauſe ACq +
10. CBq (*a*) is μν, (*c*) ſhall DFq + FEq be alſo μν. And in
c 24. 10. like manner becauſe ACB (*a*) is μν, (*d*) alſo DFE is μν.
d 24. 10. Laſtly, becauſe ACq + CBq ⊔ ACB, (*e*) ſhall DFq +
e 14. 10. FEq be ⊔ DFE. (*f*) From whence it follows that DE
f 42. 10. contains in power 2 μα. *Which was to be demonſtrated.*

## PROP. LXXII. Plate V. *Fig.* 1.

*If a rational-rectangle* A *and a medial* B, *are compounded, four irrational-lines will be made; either a binomial, or a firſt bimedial or a Major, or a line containing in power a rational and a medial-rectangle.*

Namely, If Hq = A + B, then H ſhall be one of the
four lines which the Theorem mentions. For upon
a *cor.* 16 6. CD the propounded ῥ, (*a*) make the rectangle CE = A,
b 2. *ax.* 1. and FI=B (*b*) and ſo CI=Hq. Whereas then is A ῥν,
c 21. 10. likewiſe, CE is ῥν. (*c*) therefore the latitude CF is ῥ ⊔
d 23. 10. CD, and becauſe B is μν, alſo FI ſhall be μν ; (*d*) there-
e 33. 10. fore FK is ῥ ⊔ CD, (*e*) therefore CF, FK are ῥ ⊔.
f 37. 10. and ſo the whole CK (*f*) is binom. wherefore is A ⊏ B,
g 1. 6. *i. e.* CE ⊏ FI, (*g*) then CF ⊏ FK. therefore if CF ⊔
h 1. *def.* 48. √ CFq — FKq ; (*h*) likewiſe CK ſhall be a 1 bin. and
10. conſequently H = √ CI (*k*) is a bin. If CF be ſuppoſed
k 55. 10. ⊔ √ CFq — FKq, (*l*) then ſhall CK be a 4 bin. where-
l 4. *def.* 48. fore H (√ CI) (*m*) is a major-line. But if A ⊐ B, (*g*)
10. then ſhall CF be ⊐ FK. conſequently if FK ⊔ √ FKq
m 58. 10. —CFq, (*n*) then ſhall CK be a 2. bin. (*o*) wherefore H
n 2. *def.* 48. is a firſt 2 μ. Laſtly if FK ⊔ √ FKq — CFq, (*p*) then
10. CK ſhall be a fifth binom. (*q*) whence H ſhall contain in
o 56. 10. power ῥν and μν. *Which was to be demonſtrated.*
p 5. *def.* 48.
10.
q 59. 10.

## PROP. LXXIII. *Fig.* 1.

*If two medial rectangles,* A, B, *incommenſurable to one another are compoſed, two remaining irrational-lines are made, either a ſecond bimedial, or a line containing in power two medial rectangles.*

Plate IV. *Facing Pag. 176.*

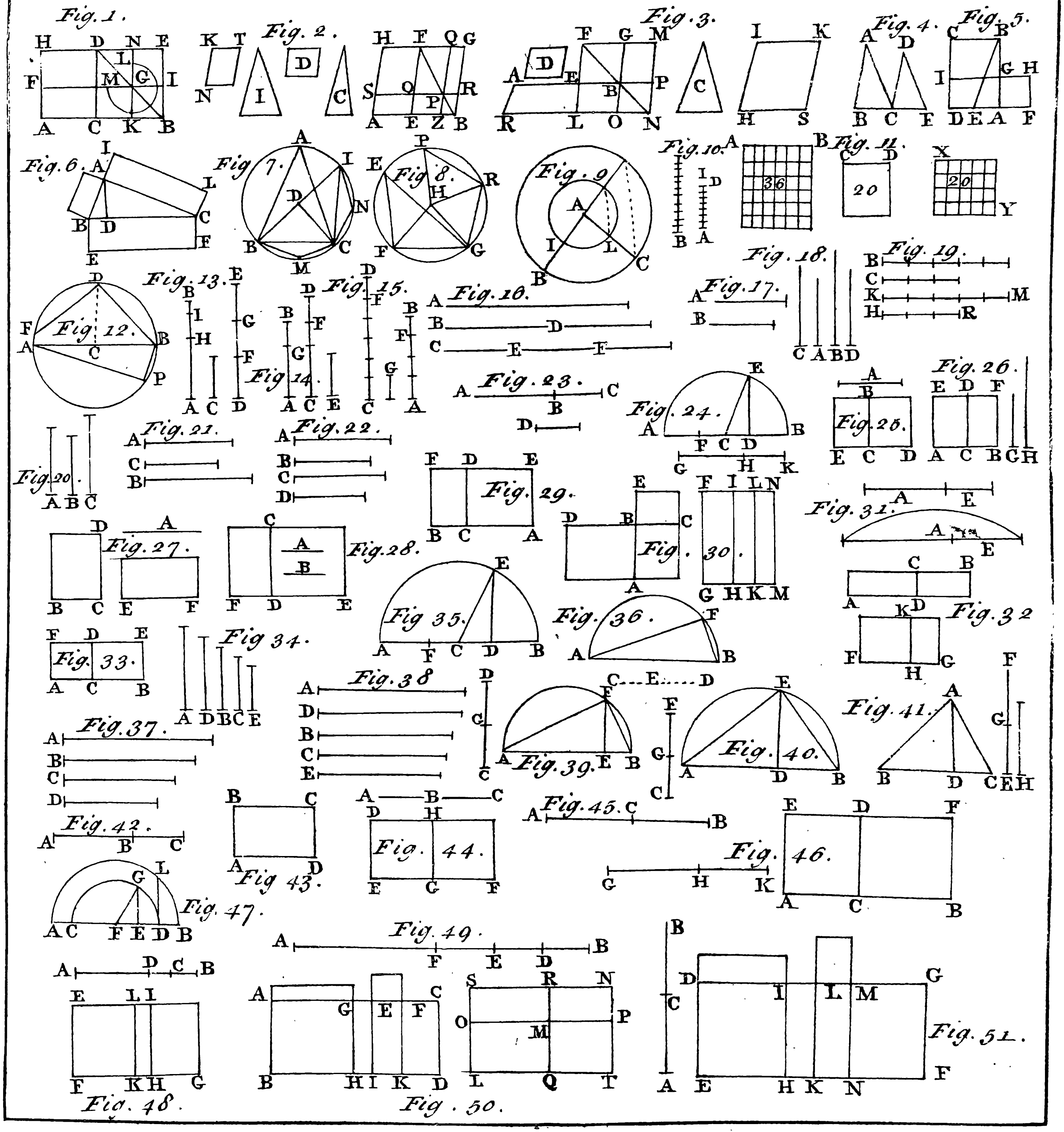


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14.
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26.
Fig. 27.
Fig. 28.
Fig. 29.
Fig. 30.
Fig. 31.
Fig. 32
Fig. 33.
Fig. 34.
Fig. 35.
Fig. 36.
Fig. 37.
Fig. 38
Fig. 39.
Fig. 40.
Fig. 41.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46.
Fig. 47.
Fig. 48.
Fig. 49.
Fig. 50.
Fig. 51.
36
20
20

As H containing in power A + B is one of the ſaid irrational lines. For upon CD propounded ρ̇ make the rectangle CE = A, and FI = B whence Hq = CI. Therefore becauſe CE and FI (*a*) are μα, (*b*) the latitudes CF, FK, ſhall be ρ̇ ⊓ CD. Alſo becauſe CE (*a*) ⊓ FI, and CE : FI (*c*) : : CF : FK, (*d*) therefore CF ⊓ FK, (*e*) therefore CK is a bin. namely, if CF ⊓ √ CFq − FKq, whence H = √ CI (*f*) ſhall be 2 μ. But if CF ⊓ √ CFq − FKq, (*g*) then CK ſhall be a 6 binom. (*h*) and conſequently H contains in power 2 μα. *Which was to be demonſtrated.*

a *hyp*
b 23. 10.
c 1. 6.
d 10. 10.
e 3. *def.* 48. 10
f 57. 10.
g 6 *def.* 48. 10.
h 60. 10.

## *Here begins the Senaires of lines irrational by Subtraction.*

### PROP. LXXIV.

D———E———F

*If from a rational-line* DF *a rational-line* DE, *commenſurable in power only to the whole* DF, *be taken away, the reſidue* EF *is irrational, and is called an Apotome or reſidual-line.*

For EFq (*a*) ⊓ DEq ; (*b*) but DEq is ρ̇ν ; (*c*) therefore EF is ρ̇. *Which was to be demonſtrated.*

In numbers ; let there be DF, 2. DE, √ 3. then EF ſhall be 2 − √ 3.

a *lem.* 26. 10.
b *hyp.*
c 10 & 11. *def.* 10.

### PROP. LXXV.

D———E———F

*If from a medial-line* DF *a medial-line* DE *commenſurable only in power to the whole* DF, *and comprehending with the whole* DF *a rational rectangle, be taken away, the remainder* EF *is irrational, and is called a firſt reſidual-line of a medial.*

For EFq (*a*) ⊓ to the rectangle FDE. therefore ſeeing FDE (*b*) is ρ̇ν. (*c*) EF ſhall be ρ̇. *Which was to be demonſtrated.*

In numbers, let DF be *v* √ 54, and DE *v* √ 24, therefore EF is *v* √ 54 − *v* √ 24.

a *ſch.* 26. 10.
b *hyp.*
c 20. *and* 11 *def.* 10.

### PROP. LXXVI.

D———E———F

*If from a medial-line* DF, *a medial-line* DE, *be taken away being commenſurable only in power to the whole* DF, *and com-*

*prehending together with the whole line* DF *a medial rectangle, the remainder* EF *is irrational, and is called a ſecond reſidual of a medial-line.*

a *hyp.* Becauſe DFq and DEq (*a*) are μν ⊏⊐, (*b*) therefore
b 16. 10. ſhall DFq + DEq be ⊏⊐ DEq; (*c*) wherefore DFq +
c 24. 10. DEq is μν, Alſo the rectangle FDE, and ſo 2 FDF, (*a*)
d *cor.* 7. 2. is μν, therefore EFq (*d*) (DFq + DEq — 2 FDE) (*e*) is
e 27. 10. ῥν. wherefore EF is ῥ. *Which was to be demonſtrated.*

In numbers, let DF be *v* √ 18. and DE *v* √ 8. then EF *v* √ 18 — *v* √ 8.

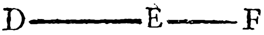
## PROP. LXXVII.

*If from a right-line* AC *be taken away a right-line* AB *being incommenſurable in power to the whole line* AC, *and making with the whole* AC *that which is compoſed of their ſquares rational, and the rectangle contained under them medial, the remainder* BC *is irrational, and is called a Minor-line.*

a *hyp.* For ACq + ABq (*a*) it ῥν. but the rectangle ACB (*a*) is
b *ſch.* 12. 10. μν; (*b*) therefore 2 CAB ⊏⊐ ACq + ABq (*c*) (2 CAB +
c 7. 2. BCq.) (*d*) therefore ACq + ABq ⊏⊐ BCq, (*e*) therefore
d 17. 10. BC is ῥ. *Which was to be demonſtrated.*

e 11. *def.* 10. In numbers, let AC be √ : 18 + √ 108; AB √ : 18 — √ 108. then BC is √ : 18 + √ : 108 — √ : 18 — √ 108.

## PROP. LXXVIII.



*If from a right-line* DF *be taken away a right-line* DE, *being incommenſurable in power to the whole line* DF, *and with the whole* DF *making that which is compoſed of their ſquares medial, and the rectangle contained under the ſame lines rational, the line remaining* EF *is irrational, and is called a line making a whole ſpace medial with a rational ſpace.*

a *hyp. and ſch.* 12. 10. For 2 FDE (*a*) is ῥν, (*b*) and DFq + DEq is μν (*c*)
b *hyp.* therefore 2 FDE ⊏⊐ DFq + DEq (*d*) (2 FDE + EFq)
c *ſch.* 12. 10. (*e*) therefore EF is ῥ. *Which was to be demonſtrated..*

d 7. 2. In numbers, let DF be $\sqrt{}^{4}$ 216 + √ 72; DE 216 — √ 72. therefore EF is $\sqrt{}^{4}$ 216 + √ 72 — $\sqrt{}^{4}$ 216 — √ 72

e *ſch.* 12. 10. *and* 11. *def.* 10.

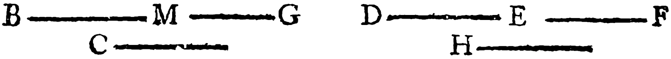
## PROP. LXXIX.

*If from a right-line* DF *be taken away a right-line* DE, *incommenſurable in power to the whole* DF, *and which together with the whole makes that which is compoſed of their ſquares medial, and the rectangle contained under them alſo medial and incommenſurable to that which is compoſed of their ſquares, the remainder is irrational, and is called a line making a whole ſpace medial with a medial ſpace.*

For 2 FDE, and FDq+DEq (*a*) are $\mu$· ; (*b*) therefore EFq (*c*) (DFq + DEq — 2 FDE) is $\dot{\rho}$. (*d*) and ſo conſequently EF is $\dot{\rho}$. *Which was to be demonſtrated.* a *hyp* & 24. 10. b 27 10. c *cor.* 7. 2 p 11 *def.* 10

In numbers; let DF be $\sqrt{}^4\ 180 + \sqrt{}\ 60$. DE $\sqrt{}^4\ 180 - \sqrt{}\ 60$, then EF ſhall be $\sqrt{}^4\ 180 + \sqrt{}\ 60 - \sqrt{}^4\ 180 - \sqrt{}\ 60$.

### *Lemma.*



*If there be the ſame exceſs between the firſt magnitude* BG *and the ſecond* C (MG) *as is between the third magnitude* DF *and the fourth* H (EF ;) *then alternately, the ſame exceſs ſhall be between the firſt magnitude* BG *and the third* DF, *as is between the ſecond* C *and the fourth* H.

For becauſe that (*a*) to the equals BM, DE, are added the equals MG, EF, that is, C, H ; the exceſs of the wholes BG, DF, (*b*) ſhall be equal to the exceſs of the parts added C, H. *Which was to be demonſtrated.* a *hyp.* b 15. *ax.* 1.

### *Coroll.*

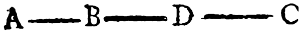
Hence, Four magnitudes Arithmetically proportional, are alternately alſo Arithmetically proportional.

## PROP. LXXX.

*To an Apotome or reſidual line* AB *only one rational right-line* BC, *being commenſurable in power only to the whole* AB, *is congruent, or can be joyned.*

If it be poſſible let ſome other line BD be added to
a 22. 10. it ; (*a*) then the rectangles ACB, ADB, (*b*) and ſo conſe-
b 22. 10. quently the doubles of them are μα ; wherefore ſeeing
c *cor.* 7. 2. ACq + BCq − 2 ACB (*c*) = ABq (*c*) = ADq + DBq − 2
d *lem.* 79. 10. ADB, therefore alternately ACq + BCq ÷ ADq +
e *hyp. and* 27. BDq (*d*) = 2 ACB ÷ 2 ADB. But ACq + BCq ÷
10. ADq + BDq (*e*) it ῥν (*f*) therefore 2 ACB ÷ 2 ADB ῥ.
f *ſch.* 12. 10. *Which is abſurd.*
g 27. 10.

## P R O P. LXXXI.



*To a firſt medial reſidual-line* AB *only one medial right-line* BC, *being commenſurable only in power to the whole, and comprehending with the whole line a rational-rectangle, can be joyned.*

Conceive BD to be ſuch a line as may be joyned
to it ; then becauſe ACq and BCq, as well as ADq and
a *hyp.* BDq (*a*) are μα ⊔ ; (*b*) alſo ACq + BCq, and ADq +
b 16. *and* 24. BDq ſhall be μα ; (*c*) but the rectangles ACB, ADB, (*d*)
10. and ſo 2 ACB and 2 ADB are ῥα ; (*e*) therefore 2 ACB
c *hyp*, ÷ 2 ADB, (*f*) that is, ACq + BCq ÷ ADq + BDq is
d *ſch.* 12. 10. ῥν. (*g*) *Which is abſurd.*
e *ſch.* 27. 10.
f 7. 2. *and lem* 79. 10.

## P R O P. LXXXII. Plate V. *Fig* 2.

g 27. 10. *Upon a ſecond medial-reſidual-line* AB *only one medial right-line* BC, *commenſurable only in power to the whole, and with it containing a medial-rectangle, can be joyned.*

If it be poſſible, let ſome other line BD be added to
it ; and upon EF ῥ make the rectangle EG = ACq
+ BCq ; as alſo the rectangle EL = ADq + BDq.
likewiſe EI = ABq. Now 2 ACB + ABq = ACq +
a 4. 2. *and* 3. BCq = EG ; therefore ſeeing EI = ABq, (*a*) alſo KG
*ax.* 1. ſhall be = 2 ACB, moreover ACq and BCq (*b*) are μα
b *hyp.* ⊔ (*c*) therefore EG (ACq + BCq) is μν (*d*) therefore
c 24. 10. the breadth EH is ῥ ⊔ EF. (*e*) Further, the rectangle
d 23. 10. ACB (*f*) and ſo 2 ACB (KG) is μν ; (*d*) therefore KH is
e *hyp.* alſo ῥ ⊔ EF. Laſtly, becauſe ACq + BCq (EG) (*g*) ⊔
f 24. 10. 2 ACB (KG) and EG : KG : : EH : KH ; (*k*) therefore
g *lem.* 26. 10. EH ⊔ HK ; (*l*) therefore EK is a reſidual-line, whereto
h 1. 6. KH is congruent, by the ſame reaſon alſo ſhall KM be
k 10. 10. congruent to the ſaid EK. *Which is repugnant to the* 80.
l 74. 10. prop. *of this Book.*

P R O P.

## PROP. LXXXIII.

*To a Major-line* AB *only* A——B———D——C
*one right-line* BC *can be joyned being incommenſurable in power to the whole, and making together with the whole line that which is compoſed of their ſquares rational, and the rectangle which is contained under them medial.*

Conceive any other BD to be congruent to it; Therefore whereas. ACq + BCq, and ADq + BDq (*a*) are ρα, a 1. *hyp.*
their exceſs (2 (*b*) ACB ÷ 2 ADB) (*c*) is ρν. *Which is* b *lem.* 97.
*abſurd*; becauſe ACB and ADB are μα by the Hyp. 10.
c *ſch.*27.10
d 27. 10.

## PROP. LXXXIV.

*Unto a line* (AB) *making* A——B———D——C
*with a rational-ſpace a whole ſpace medial only one right-line* BC *can be joined, being incommenſurable in power to the whole, and making together with the whole that which is compoſed of their ſquares medial, and the rectangle which is contained under them rational.*

Suppoſe ſome other BD be congruent alſo to it; (*a*)
then the rectangles ACB, ADB, (*b*) and ſo 2 ACB and 2 a *hyp.*
ADB are ρα; therefore 2 ACB ÷ 2 ADB, (*c*) that is, b *ſch.* 12.
ACq + BCq ÷ ADq + BDq (*d*) is ρν. *Which is abſurd*; 10.
ſince ACq + BCq, and ADq + BDq are μα by the c *lem.* 79.
Hyp. 10.

## PROP. LXXXV. Plate V. *Fig.* 3.

d *ſch.*27.10

*To a line* AB, *which with a medial-ſpace makes a whole ſpace medial, can be joined only one right-line* BC, *incommenſurable in power to the whole, and making with the whole both that which is compoſed of their ſquares medial, and the rectangle which is contained under them medial and incommenſurable to that which is compoſed of their ſquares.*

Thoſe things being ſuppoſed which are done and ſhewn in the 82. *prop.* of this Book; it is clear that EH and KH are ρ ⊔ EF. Beſides, ſince, ACq + CBq,
that is, the rectangle EG, (*a*) is ⊔ ACB, (*b*) and ſo EG a *hyp.*
⊔ 2 ACB (KG;) and EG : KG (*c*) : : EH : KH, ſhall EH b 14. 10.
be ⊔ KH; therefore EH is a reſidual-line, and the c 1. 6.
line congruent to it is KH. In like manner may KM be ſhewn to be congruent to the reſidal EK, againſt the 80. *prop.* of this Book.

## *Third Definitions.*

A Rational-line and a reſidual being propounded, if the whole be more in power than the line joined to the reſidual, by the ſquare of a right-line commenſurable unto it in length; then

I. If the whole be commenſurable in length to the rational-line propounded, it is called a firſt reſidual-line.

II. But if the line adjoined be commenſurable in length to the rational-line propounded, it is called a ſecond reſidual-line.

III. If neither the whole nor the line adjoined be commenſurable in length to the rational-line propounded, it is called a third reſidual-line.

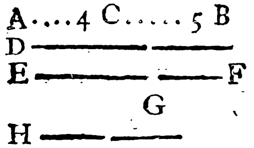
Moreover, if the whole be more in power than the line adjoined by the ſquare of a right-line incommenſurable to it in length; then

IV. If the whole be commenſurable in length to the rational-line propounded, it is called a fourth reſidual-line.

V. But if the line adjoined be commenſurable in length to the rational-line propounded, it is a fifth reſidual.

VI. If neither the whole nor the line adjoined be commenſurable in length to the rational-line propounded, it is termed a ſixth reſidual-line.

## PROP. LXXXVI, 87, 88, 89, 90, 91.



*To find out a firſt, ſecond, third, fourth, fifth, and ſixth reſidual-line.*

Reſidual-lines are found out by ſubducting the leſs names or parts of binomials from the greater. Ex. gr. Let $6 + \sqrt{20}$ be a firſt binomial, then ſhall $6 - \sqrt{20}$ be a firſt reſidual. So that it is not neceſſary to repeat more concerning the finding of them out.

### *Lemma.* Plate V. *Fig.* 4, 5.

*Let* AC *be a rectangle contained under the right-lines* AB, AD. *Let* AD *be drawn forth to* E, *and* DE *equally divided in* F; *and let the rectangle* AGE *be* = FEq, *and the rectangles* AI, DK, FH, *finiſhed. Then let the ſquare* LM = AH *be made, and the ſquare* NO = GI; *and the lines* NSR, OST, *produced.*

I ſay, 1. The rectangle AI = LM + NO = TOq + SOq, which appears by the conſtr.

2. *The rectangle* DK = LO. For becauſe the rectangle AGE (*a*) = FEq (*b*) thence are AG, FE, GE ∺ (*c*) and ſo AH, FI, GI ∺, (*a*) that is, LM, FI, NO ∺; but LM, LO, NO (*d*) are ∺; therefore FI = (*e*) LO (*f*) = DK = (*g*) NM.

3. *Hence,* AC = AI − DK − FI = LM + NO − LO − NM = TR.

4. *It is manifeſt that* DF, FE, DE, *are* ⊐⊏.

5. *If* AE ⊐⊏ DE, *and* AE ⊐⊏ $\sqrt{\text{AEq} - \text{DEq}}$ (k) *then ſhall* AG, GE, AE *be* ⊐⊏.

6. *Alſo, becauſe* AE (l) ⌐⊐⊏ DE, (m) *thence ſhall* AE, FE, *be* ⌐⊐⊏; (n) *and ſo* AI, FI, *that is,* LM + NO *and* LO *are* ⌐⊐⊏.

7. *Becauſe* AG* ⊐⊏ GE, (n) *ſhall* AH, GI, *that is,* LM, NO *be* ⊐⊏.

8. *But beeauſe* AE (*l*) ⌐⊐⊏ DE, (o) *therefore ſhall* FE, GE *be* ⌐⊐⊏, (n) *and ſo the rectangle* FI ⌐⊐⊏ GI, *that is,* LO ⌐⊐⊏ NO; *wherefore ſeeing* LO : NO (p) : : TS : SO; (q) *therefore ſhall* TS, SO *be* ⌐⊐⊏.

9. *If* AE *be put* ⌐⊐⊏ $\sqrt{\text{AEq} - \text{DEq}}$, *then ſhall* AG, GE, AE *be* ⌐⊐⊏.

10. (*ſ*) *Wherefore the rectangles* AH, GI, *that is,* TOq, SOq *ſhall be* ⌐⊐⊏.

a *conſtr.*
b 17. 6.
c 1. 6.
d *ſch.* 22. 6.
e 9. 5.
f 36. 1.
g 43. 1.
h 16. 10.
k 18. *and* 10. 10.
l *hyp.*
m 13. 10.
n 1. 6. *and* 10. 10.
* *before.*
o 14. 10.
p 2. 6.
q 10. 10.
r 19. 10. & 17. 10.
ſ 1. 6. *and* 10. 10.

### PROP. XCII. *Fig.* 4, 5.

*If a ſpace* AC *be contained under a rational-line* AB, *and a firſt reſidual-line* AD (AE − DE) *the right-line* TS, *which containeth the ſpace* AC *in power, is a reſidual-line.*

a *hyp.*
b 12. 10.
c 20. 10.
d *lem.* 91. 10.
e 74. 10.

Uſe the foregoing *Lemma* for a preparation to the demonſtration of this prop. Therefore TS = √ AC. Alſo AG, GE, AE, are ⊐⊏; therefore ſince AE ⊐⊏ (*a*) AB ῥ, (*b*) alſo AG and GE ſhall be ⊐⊏ AB, (*c*) therefore the rectangles AH and GI, that is, TOq and SOq are ῥα. (*d*) Likewiſe TO, SO, are ῥ ⊐⊑ (*e*) and conſequently TS is a reſidual-line. *Which was to be demonſtrated.*

## PROP. XCIII. Plate V. *Fig.* 4, 5.

*If a ſpace* AC *be contained under a rational line* AB, *and a ſecond reſidual* AD (AE — DE) *the right-line* TS, *containing the ſpace* AC *in power, is a firſt medial reſidual-line.*

a *hyp.*
b 13. 10.
c 22. 10.
d *lem.* 74. 10.
e *hyp.*
f 20. 10.
g 75. 10.

Again, by the foregoing *Lemma*, AG, GE, AE are ⊐⊏; therefore (*a*) ſince AE is ῥ ⊐⊏ AB, (*b*) alſo AG, GE, ſhall be ῥ ⊐⊏ AB, (*c*) therefore the rectangles AH, GI, that is, TOq, SOq are μα; (*d*) likewiſe TO ⊐⊑ SO. Laſtly, becauſe DE (*e*) ⊐⊏ AB ῥ, (*f*) the right-angle DI, and the half thereof DK or LO, that is, TOS ſhall be ῥν; (*g*) from whence it follows that TS (√ AC) is a firſt medial reſidual. *Which was to be domonſtrated.*

## PROP. XCIV. *Fig.* 4, 5.

*If a ſpace* AC *be contained under a rational-line* AB *and a third reſidual* AD (AE — DE) *the right-line* TS *containing in power the ſpace* AC *is a ſecond medial reſidual-line.*

a *hyp.*
b 22. 10.
c 24. 10.
d 76. 10.

As in the former, TO and SO are μ. Therefore becauſe DE (*a*) is ῥ ⊐⊏ AB, (*b*) the rectangle DI, (*c*) and ſo DK, or TOS, ſhall be μν; therefore TS = √ AC is a ſecond medial reſidual. *Which was to be demonſtrated.*

## PROP. XCV. *Fig.* 4, 5.

*If a ſpace* AC *be contained under a rational-line* AB *and a fourth reſidual* AD (AE — DE) *the right-line* TS *containing the ſpace* AC *in power, is a Minor-line.*

a *lem.* 91. 10.
b *hyp.*
c 20. 10.
d 77. 10.

As before, TO (*a*) ⊐⊏ SO. Therefore becauſe AE (*b*) is ῥ ⊐⊏ AB, *c*) ſhall AI (TOq + SOq) be ῥν, but, as before, the rectangle TOS is μν; (*d*) therefore TS = √ AC is a *Minor*-line. *Which was to be demonſtrated.*

## PROP. XCVI. Plate V. *Fig.* 4 5.

*If a ſpace* AC *be contained under a rational-line* AB *and a fifth reſidual* AD (AE -DE) *the right-line* TS *containing in power the ſpace* AC, *is a line which maketh with a rational ſpace the whole ſpace medial.*

For again TO ⌐⊔ SO. Therefore ſince AE (*a*) is ῥ ↽⊔ AB (*b*) alſo AT, that is, TOq + SOq ſhall be μ. But, as in 93. the rectangle TOS is ῥν; (*c*) whence TS = √ AC is a line which with ῥν makes a whole μν. *Which was to be demonſtrated.*

a *hyp.*
b 22. 10.
c 78. 10.

## PROP. XCVII. *Fig.* 4, 5.

*If a ſpace* AB *contained under a rational-line* AB, *and a ſixth reſidual* AD (AE – DE) *the right-line* TS *containing in power the ſpace* AC *is a line making with a medial rectangle, a whole ſpace medial.*

As often above TO ↽⊔ SO. Alſo, as in 96. TOq + SOq is μν, but the rectangle TOS is ῥν, as in 94. (*a*) Laſtly, TOq + SOq ↽⊔ TOS (*b*) therefore, TS = √ AC is a line which with μ. makes a whole μν. *Which was to be dem.*

a *lem.* 91. 10.
b 79. 10.

### *Lemma.* *Fig.* 6.

*Upon a right-line* DE * *apply the rectangles* DF = ABq, *and* DH = ACq, *and* IK = BCq. *and let* GL *be biſected in* M, *and the line* MN *drawn parallel to* GF.

* *cor.* 16. 6.

Then 1. *The rectangle* DK *is* = ACq + BCq. *as the conſtruction manifeſts.*

2. *The rectangle* ACB = GN or MK. For DK (*a*) = ACq + BCq (*b*) = 2 ACB + ABq; but ABq (*a*) = DF. therefore GK (*c*) = 2 ACB; and conſequently GN or MK = ACB.

3. *The rectangle* DIL = MLq. For becauſe ACq : ACB (*e*) : : ACB : BCq, that is, DH : MK : : MK : IK. (*e*) thence is DI : ML : : ML : IL (*f*) therefore DIL = MLq.

a *conſtr.*
b 7. 2.
c 3. *ax.* 1.
d 7. *ax.* 1.
e 1. 6.
f 17. 6.

4. *If* AC *be taken* ⌐⊔ BC, *then* DK *ſhall be* ⌐⊔ ACq. For ACq + BCq (DK) (*g*) ⌐⊔ ACq.

g 16. 10.

5. *Likewiſe* DL ⌐⊔ $\sqrt{\text{DLq} - \text{GLq}}$. For becauſe DH (ACq) ⌐⊔ IK (BCq) (*h*) thence ſhall DI be ⌐⊔ IL; (*k*) therefore $\sqrt{\text{DLq} - \text{GLq}}$ ⌐⊔ DL.

6. *Alſo* DL ↽⊔ GL. For ACq + BCq ↽⊔ (*l*) 2 ACB. That is, DK ↽⊔ GK; (*m*) therefore DL ↽⊔ GL.

h 10. 10.
k 18. 10.
l *lem.* 26. 10
m 10. 10.

7. *But if* AC *be taken* ˥ɟ BC, *then* DL ſhall be ˥⊔ $\sqrt{DLq - GLq}$.

### PROP. XCVIII. Plate V. *Fig*. 6.

a *hyp.*
b *lem.* 97. 10.
c *ſch.* 12. 10.
d 21. 10.
e 22. *and* 24. 10.
f 23. 10.
g 13. 10.
h *ſch.* 12. 10.
k 74. 10.
l 1 *def.* 85. 10.
m *lem.* 97. 10.

*The ſquare of a reſidual-line* AB (AC—BC) *applyed to a rational-line* DE, *makes the breadth* DG *a firſt reſidual-line.*

Do as is enjoined in the *Lemma* next preceding. Then becauſe AC, BC, (*a*) are ρ̊ ˥ɟ; (*b*) alſo DK (ACq+BCq) ſhall be ˥⊔ ACq. (*c*) Therefore DK is ρν; (*d*) wherefore DL is ρ̊ ˥⊔ DE. (*e*) Likewiſe the rectangle GK (2 ACB) is μν; (*f*) therefore GL is ρ̊ ˥⊔ DE, (*g*) and conſequently DL ˥⊔ GL. (*h*) But DLq ˥⊔ GLq, (*k*) therefore DG is a reſidual, (*l*) and that of the firſt order (becauſe (*m*) AC ˥ɟ BC, and therefore DL ˥⊔ $\sqrt{DLq - GLq}$.) *Which was to be demonſtrated.*

### PROP. XCIX. *Fig*. 6.

a *hyp.*
b *lem.* 97. 10.
c 24. 10.
d 23. 10.
e *hyp.* *and* *ſch.* 12. 10.
f 21. 10.
g 13. 10.
h *ſch.* 12. 10.
k 74. 10.
l *lem.* 97. 10.
m 2. *def.* 85. 10.

*The ſquare of a firſt medial reſidual-line* AB (AC—BC) *applied to a rational-line* DE, *makes the breadth* DG *a ſecond reſidual-line.*

Suppoſing the foregoing *Lemma*; becauſe AC and BC (*a*) are μ. ˥⊔, (*b*) thence ſhall DK (ACq+BCq) be ˥⊔ ACq; (*c*) wherefore DK is μν; (*d*) therefore DL is ρ̊ ˥⊔ DE; (*e*) alſo GK (2 ACB) is ρ̊; (*f*) therefore GL is ρ̊ ˥⊔ DE; (*g*) wherefore DL ˥⊔ GL. (*h*) But DLq ˥⊔ GLq; (*k*) therefore DG is a reſidual-line. And becauſe DL is ˥⊔ $\sqrt{DLq - GLq}$, (*m*) therefore ſhall DG be a ſecond reſidual. *Which was to be demonſtrated.*

### PROP. C. *Fig*. 6.

a 23. 10.
b *lem.* 26. 10.
c 1. 6. *and* 10. 10.
d *ſch.* 12. 10.
e 74. 10.
f 3. *def.* 85. 10.
g *lem.* 97. 10.

*The ſquare of a ſecond medial reſidual-line* AB (AC—BC) *applied to a rational-line* DE, *makes the breadth* DG *a third reſidual-line.*

Again DK is μν. (*a*) wherefore DL is ρ̊ ˥⊔ DE; alſo GK is μν, (*a*) whence GL is ρ̊ ˥⊔ DE; (*b*) likewiſe DK ˥⊔ GK; (*c*) wherefore DL ˥⊔ GL; (*d*) but DLq ˥⊔ GLq. (*e*) therefore DG is a reſidual-line, and that of (*f*) the third order, (*g*) becauſe DL ˥⊔ $\sqrt{DLq - GLq}$. *Which was to be demonſtrated.*

## PROP. CI. Plate V. *Fig.* 6.

*The ſquare of a Minor-line* AB (AC—BC) *applied to a rational-line* DE, *makes the breadth* DG *a fourth reſidual.*

As before, ACq + BCq, that is DK, is ρν. (*a*) therefore DL is ρ ⊤⊔ DE. but the rectangle ACB, and ſo GK (2 ACB) * is μν. (*b*) wherefore GL is ρ ⊤⊔ DE. (*c*) therefore DL ⊤⊔ GL. (*d*) but DLq ⊤⊔ GLq. and becauſe * ACq ⊤⊔ BCq, (*e*) thence ſhall DL be ⊤⊔ √ DLq — GLq. (*f*) therefore DG has the conditions required to a fourth reſidual. *Which was to be demonſtrated.*

a 21. 10.
* *hyp.*
b 23. 10.
c 13. 10.
d *ſch.* 12. 10.
e *lem.* 97. 10.
f 4. *def.* 85. 10.

## PROP. CII. *Fig.* 6.

*The ſquare of a line* AB (AC—BC) *which makes with a rational ſpace the whole ſpace medial, applied to a rational-line* DE, *makes the breadth* DG *a fifth reſidual-line.*

For, as above, DK is μν. (*a*) wherefore DL is ρ ⊤⊔ DE alſo GK is ρν. (*b*) whence GL is ρ ⊤⊔ DE. (*c*) therefore DL ⊤⊔ GL. (*d*) but DLq ⊤⊔ GLq. Moreover DL (*e*) ⊤⊔ √ DLq—GLq. wherefore DG (*f*) is a fifth reſidual. *Which was to be demonſtrated.*

a 23. 10.
b 21. 10.
c 13. 10.
d *ſch.* 12. 10.
e *lem.* 97. 10.
f 5. *def.* 85. 10.

## PROP. CIII. *Fig.* 6.

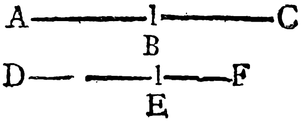
*The ſquare of a line* AB (AC — BC) *making with a medial ſpace the whole ſpace medial, applied to a rational-line* DE, *makes the breadth* DG *a ſixth reſidual-line.*

As above, DK and GK are μα; (*a*) wherefore DL and GL are ρ ⊤⊔ DE. alſo DK (*b*) ⊤⊔ GK. (*c*) whence DL ⊤⊔ GL. (*d*) therefore DG is a reſidual. (*b*) And whereas ACq ⊤⊔ BCq. and ſo DL ⊤⊔ √ DLq − GLq, (*e*) therefore DG ſhall be a ſixth reſidual. *Which was to be demonſtrated.*

a 23. 10.
b *hyp.* and *lem.* 97. 10.
c 10. 10.
d 74. 10.
e 6. *def.* 85. 10.

## PROP. CIV.

*A right-line* DE *commenſurable in length to a reſidual* AB (AC—BC) *is it ſelf alſo a reſidual, and of the ſame order.*



*Lemma.*

*Let* AB : DE : : AC : DF, *and* AB ⊤⊔ DE.

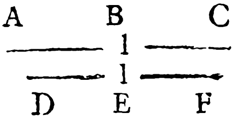
I ſay AC + BC ⌐⊔ DF + EF. For AC : BC (a) : : DF : EF. therefore by compounding AC + BC : BC : : DF + EF : EF therefore by permutation AC + BC : DF + EF : : BC : EF. (a) but BC ⌐⊔ EF; (b) therefore AC + BC ⌐⊔ DF + EF, *Which was to be dem.*

a *lem.* 66. 10.
b 10. 10.

(a) Make AB : DE : : AC : DF, (b) therefore AC + BC ⌐⊔ DF + EF. therefore ſeeing AC + BC (c) is a binomial, (d) DF + EF ſhall be a binomial too, and of the ſame order. (e) wherefore DF — EF is a reſidual of the ſame order with AC — BC. *Which was to be demonſtrated.*

a 12. 6.
b *lem.* 103. 10
c *hyp.*
d 67. 10.
e *by def.* 85. 10.

## PROP. CV.

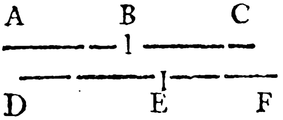


*A right-line* DE *commenſurable to a medial reſidual-line* AB (AC — BC) *is it ſelf a medial reſidual, and of the ſame order.*

Again (a) make AB : DE : : AC : DF : (b) whence AC + BC ⌐⊔ DF + EF. (c) therefore DF + EF is a bimedial of the ſame order with AC + BC, (d) and conſequently DF — EF ſhall be a medial reſidual of the ſame order with AC — BC. *Which was to be dem.*

a 12. 6.
b *lem.* 103. 10.
c 68. 10.
d 75. *and* 76. 10.

## PROP. CVI.

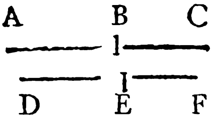


*A right-line* D E *commenſurable to a Minor-line* A B (AC — BC) *is it ſelf alſo a Minor-line.*

Make AB : DE : : AC : DF. (a) then is AC + BC ⌐⊔ DF + EF. but AC + BC (b) is a *Major* line; (c) therefore DF + EF is alſo a *Major* line; (d) and conſequently DF — EF is a *Minor* line. *Which was to be dem.*

a *lem.* 103. 10.
b *hyp.*
c 69. 10.
d 77. 10.

## PROP. CVII.



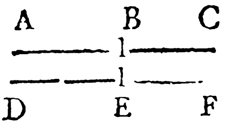
*A right-line* D E *commenſurable to a line* AB (AC — BC) *which makes with a rational ſpace the whole ſpace medial, is it ſelf alſo a line making with a rational ſpace the whole ſpace medial.*

For, accordingly as in the former, we may ſhew DF + EF to contain in power $\dot{\rho}\nu$ and $\mu.\nu.$ (a) whence DF — EF is a line making, &c.

a 78. 10.

PROP.

## PROP. CVIII.



*A right-line* DE *commenſurable to a line* AB (AC − BC) *which with a medial ſpace makes the whole ſpace medial, is it ſelf a line making with a medial ſpace the whole ſpace medial.*

For according to the preceding DF + EF ſhall contain in power 2 μα. (*a*) therefore DF—EF ſhall be, as in the Prop.

a 79. 10.

## PROP. CIX. Plate V. *Fig.* 7.

*A medial rectangle* B *being taken from a rational rectangle* A+B, *the right-line* H *which containeth in power the ſpace remaining* A, *is one of theſe two irrational-lines*, viz. *either a reſidual-line, or a Minor-line.*

Upon CD ῥ make the rectangles CI = A + B, and FI = B. whence CE (*a*) = A = Hq. wherefore becauſe CI (*b*) is ῥν. (*c*) therefore CK is ῥ $\sqcap\!\!\!\sqcup$ CD; but becauſe FI (*b*) is μν, (*d*) ſhall FK be ῥ $\ulcorner\!\sqcap\!\!\!\sqcup$ CD; (*e*) whence CK $\ulcorner\!\sqcap\!\!\!\sqcup$ FK; (*f*) therefore CF is a reſidual-line. Wherefore if CK be $\sqcap\!\!\!\sqcup$ $\sqrt{CKq - FKq}$. (*g*) then CF ſhall be a firſt reſidual; (*h*) therefore √ CE (H) is a reſidual-line. But if CK $\ulcorner\!\sqcap\!\!\!\sqcup$ $\sqrt{CKq - FKq}$ (*k*) then CF ſhall be a fifth reſidual; and conſequently H ( √ CE ) (*l*) ſhall be a *Minor*-line. *Which was to be demonſtrated.*

a 3. *ax.* 1.
b *hyp. and conſtr.*
c 21. 10.
d 23. 10.
e 13. 10.
f 74. 10.
g 1. *def.* 85. 10.
h 92. 10.
k 4. *def.* 85. 10.
l 95. 10.

## PROP. CX. *Fig.* 7.

*A rational rectangle* B *being taken away from a medial rectangle* A+B, *other two irrational-lines are made, namely; either a firſt medial reſidual-line, or a line making with a rational ſpace the whole ſpace medial.*

Upon CD the propounded ῥ make the rectangle CI = A + B, and FI = B (*a*) whence CE = A = Hq. Therefore becauſe CI (*b*) is μν; (*c*) ſhall CK be ῥ $\ulcorner\!\sqcap\!\!\!\sqcup$ CD. but becauſe FI (*b*) is ῥν (*d*) thence FK ῥ $\sqcap\!\!\!\sqcup$ CD. (*e*) whence CK $\ulcorner\!\sqcap\!\!\!\sqcup$ FK. (*f*) therefore CF is a reſidual, (*g*) and that a ſecond. If CK $\sqcap\!\!\!\sqcup$ $\sqrt{CKq - FKq}$, (*h*) then H (√ CE) is a firſt medial reſidual. But if CK $\ulcorner\!\sqcap\!\!\!\sqcup$ √ CKq — FKq, (*k*) then ſhall CF be a fifth reſidual; and (*l*) conſeqeently H (√ CE) ſhall be a line making μν with ῥν. *Which was to be demonſtrated.*

a 3. *ax.* 1.
b *hyp. and conſtr.*
c 23. 10.
d 21. 10.
e 13. 10.
f 74. 10.
g 2. *def.* 85. 10.
h 93. 10.
k 5. *def.* 85. 10.
l 96. 10.

## P R O P. CXI. Plate V. *Fig.* 7.

*A medial ſpace* B *being taken away from a medial ſpace* A + B, *which is incommenſurable to the whole* A + B, *the other two irrational-lines are made,* viz. *either a ſecond medial reſidual-line, or a line making with a medial ſpace the whole ſpace medial.*

a 3. *ax.* 1. Upon CD ῥ make the rectangles CI = A + B, and
b 23. 10. FI = B. (*a*) wherefore CE = A = Hq. Becauſe there-
c *hyp.* fore CI is μν, (*b*) thence CK is ῥ ‘⊐⊏ CD. and in like
d 10. 10. manner FK ῥ ‘⊐⊏ CD. Likewiſe becauſe CI (*c*) ‘⊐⊏ FI,
e 74. 10. (*d*) therefore CK ‘⊐⊏ FK (*e*) wherefore CF is a reſidual,
f 3. *def.* 85. 10. (*f*) namely a third, If CK ⊐⊏ $\sqrt{CKq - FKq}$, (*g*) whence
H ($\sqrt{CE}$) ſhall be a ſecond medial reſidual; but if
g 94. 10. CK ‘⊐⊏ $\sqrt{CKq - FKq}$. (*h*) then ſhall CF be a ſixth re-
h 6. *def.* 85. 10. ſidual; (*k*) wherefore A ſhall be a line making μν with
k 97. 10. μ. *Which was to be demonſtrated.*

## P R O P. CXII. *Fig.* 8.

*A reſidual-line* A *is not the ſame with a binomial-line.*

Upon BC propounded ῥ make the rectangle CD = Aq.
a 98. 10. Therefore ſeeing A is a reſidual, (*a*) BD ſhall be a firſt
reſidual, to which let DE be the line congruent or that
b 74. 10. may be adjoined; (*b*) wherefore BE, DE, are ῥ ⊐⊏; (*c*)
c 1. *def.* 85. 10. and BE ⊐⊏ BC. If you conceive A to be a binomial,
then BD is a firſt binomial, whoſe names let be BF, FD;
d 37. 10. and let BF be ⊏ FD; (*d*) therefore BF, FD are ῥ ⊐⊏;
e 1. *def.* 48. 10. and BF (*e*) ⊐⊏ BC. therefore ſince BC ⊐⊏ BE, (*f*)
ſhall BE be ⊐⊏ BF. (*g*) and thence BE ⊐⊏ FE, (*h*)
f 12, 10. therefore FE is ῥ. Likewiſe becauſe BE ‘⊐⊏ DE, (*k*) ſhall
g *cor.* 16. 10. FE be ‘⊐⊏ DE; (*l*) wherefore FD is a reſidual, and ſo
FD is ῥ. but it was ſhewn ῥ. which are repugnant;
h *ſch.* 12. 10. therefore A is falſely conceived to be a binomial.
*Which was to be demonſtrated.*

k 14. 10.
l 74. 10.

## *The names of the 13 irrational-lines differing one from another.*

1. A Medial-line.
2. A binomial-line; of which there are ſix ſpecies.
3. A firſt bimedial-line.
4. A ſecond bimedial.
5. A Major-line.
6. A line containing in power a rational ſuperficies, and a medial ſuperficies.
7. A line containing in power two medial ſuperficies.
8. A reſidual-line; of which there are alſo ſix kinds.
9. A firſt medial reſidual-line.
10. A ſecond medial reſidual-line.
11. A Minor-line.
12. A line making with a rational ſuperficies the whole ſuperficies medial.
13. A line making with a medial ſuperficies the whole ſuperficies medial.

*Since the differences of breadths argue differences of right-lines, whoſe ſquares are applied to ſome rational-line, and it is demonſtrated in the preced. Propoſitions that the breadths which ariſe from applying of the ſquares of theſe* 13 *lines to differ one from another, it evidently follows that theſe* 13 *lines do alſo differ one from another.*

### PROP. CXIII. Plate V. *Fig.* 8.

*The ſquare of a rational-line* A *applied to a binomial* BC ( BD + DC ) *makes the breadth* EC *a reſidual-line, whoſe names* EH, CH, *are commenſurable to the names* BD, DC, *of the binomial-line, and in the ſame proportion* (EH : BD : : CH : DC; ) *and moreover, the reſidual-line* EC *which is made, is of the ſame order with* BC *the binomial.*

Upon DC the leſs name (*a*) make the rectangle DF a *cor.* 16.6.
= Aq = BE, whence BC : CD (*b*) : : FC : CE. there- b 14. 6.
fore by diviſion, BD : DC : : FE : EC. And whereas
BD (*c*) $\sqsubset$ DC, (*d*) thence FE ſhall be $\sqsubset$ EC; Take EG c *hyp.*
= EC, and make FG : GE : : EC : CH. Then EH, d 14. 5.
and CH ſhall be the names of the reſidual EC, where-
unto all is agreeable that is propounded in the theorem.
For by compounding, FE : GE (EC) : : EH : CH. there- e 12. 5.
fore FH : EH (*e*) : : EH : CH (*f*) : : FE : EC (*f*) : : BD : f *before.*
DC. wherefore ſince BD (*g*) $\top$ DC (*h*) thence ſhall EH g *hyp.*
be h 10. 10.

be ⊤⊐ EH, (*h*) and FHq ⊤⊔ EHq. Therefore becauſe
k *cor.* 20 6. FHq : EHq (*k*) : FH : CH : (*l*) ſhall FH be ⊤⊔ CH ; (*l*)
l 16. 10. and ſo FC ⊤⊔ CH. Moreover CD (*g*) is ῤ, and DF (Aq)
m 21. 10. (*g*) is ῤν; (*m*) therefore FC is ῤ, ⊤⊔ CD, whence alſo CH
n *ſch.* 12. 10. is ῤ ⊤⊔ CD; (*n*) therefore EH, CH are ῤ and ⊤⊐, as before; (*o*) therefore EC is a reſidual-line, to which CH may
o 74. 10. be joined. Furthermore EH : CH (*f*) : : BD : DC, and
p 10. 10. ſo by permutation EH : BD : : CH : DC; whence be-
q 15. 10. cauſe CH (*f*) ⊤⊔ DC, (*p*) ſhall EH be ⊤⊔ BD. But
r 12. 10. ſuppoſe BD ⊤⊔ $\sqrt{BDq - DCq}$, (*q*) then ſhall EH be
ſ 1. *def.* 48. 10. ⊤⊔ $\sqrt{EHq. - CHq}$. Alſo if BD ⊤⊔ ῤ propounded,
then ſhall EH be ⊤⊔ to the ſame ῤ. (*ſ*) that is, if BC be
t 1. *def.* 85. 10. a firſt binomial, (*t*) EC ſhall be a firſt reſidual. In like
manner, if DC be to the ⊤⊔ propounded ῤ, (*r*) then is
u 2. *def.* 48. 10. CH ⊤⊔ to the ſame ῤ. (*u*) that is, if BC be a ſecond
binomial, (*x*) EC ſhall be a ſecond reſidual : And if
x 2. *def.* 85. 10. this be a third binomial, then that ſhall be a third reſidual, *&c.* But if BD be ⊤⊔ $\sqrt{BDq - DCq}$, (*y*) then
y 15. 10. ſhall EH be ⊤⊔ $\sqrt{EHq - CHq}$. therefore if BC be a
4, 5, or 6 binomial, EG ſhall be likewiſe a 4, 5, or 6, reſidual. *Which was to be demonſtrated.*

## P R O P. CXIV. Plate V. *Fig.* 9.

*The ſquare of a rational line* A *applied to a reſidual-line* BC (BD — CD) *makes the breadth* BE *a binomial; whoſe names* BE, GE *are commenſurable to the names* BD, BC *of the reſidual-line* LC, *and in the ſame proportion, and moreover, the binomial-line which is made* (BE) *is of the ſame order with the reſidual-line* (BC.)

a *cor.* 16. 6. (*a*) Make the rectangle DF = Aq. and BF : FE (*b*)
b 12. 6. : : EG : GF; whence for that DF=Aq=CE, (*c*) there-
c 14. 6. fore BD : BC : : BE : BF. therefore by converſion of pro-
d 19. 5. portion BD : CD : : BE : FE : : EG : GF : : (*d*) GB : EG.
e *hyp.* but BD (*e*) ⊤⊐ CD; (*f*) therefore BG ⊤⊐ GE. therefore
f 10. 10. becauſe BGq : GEq (*g*) : : BG : GF. (*h*) ſhall BG be ⊤⊔ GF;
g *cor.* 20. 6. (*k*) and ſo BG ⊤⊔ BF. moreover BD (*e*) is ῤ, and the rec-
h 10. 10. tangle DF (Aq) (*e*) is ῤν. (*l*) therefore BF is ῤ ⊤⊔ BD. (*m*)
k *cor.* 16. 10. therefore alſo BG is ῤ ⊤⊔ BD. (*n*) therefore BG, GE
l 21. 20. are ῤ ⊤⊐. (*o*) wherefore BE is a binomial. Laſtly, be-
m 12. 10. cauſe BD : CD : : BG : GE. and by permutation BD : BG
n *ſch.* 12. 10. : : CD : GE, and BD ⊤⊔ BG (*p*) thence ſhall CD be ⊤⊔
o 37. 10. GE. therefore if CB be a firſt reſidual, BE ſhall be a firſt
p 10. 10. binomial, *&c.* as in the *prec.* therefore, *&c.*

P R O P.

## PROP. CXV. Plate V. *Fig.* 10.

*If a ſpace* AB *be contained under a reſidual-line* AC (CE — AE) *and a binomial* CB, *whoſe names* CD, DB, *are commenſurable to the names* CE, AE, *of the reſidual-line, and in the ſame proportion* (CE : AE : : CD : DB.) *then the right-line* F *which containeth in power that ſpace* AB, *is rational.*

Let G be $\acute{\rho}$. and make the rectangle CH = Gq ; (*a*) a 113. 10.
then ſhall BH (HI—IB) be a reſidual-line, and HI (*a*) b *hyp.*
⊤⊥ CD (*b*) ⊤⊥ CE. (*a*) and BI ⊤⊥ DB. (*a*) and HI : c 19. 5.
BI : : CD : DB (*b*) : : CE : EA. therefore by permutation d 12. 10.
HI : CE : : BI : EA. (*c*) therefore BH : AC : : HI : CE e 10. 10.
: : BI : EA. wherefore ſince (*d*) HI ⊤⊥ CE, (*e*) thence f 1. 6. *and*
BH ⊤⊥ AC. (*f*) therefore the rectangle HC ⊤⊥ BA. 10. 10.
But HC (Gq) (*b*) is $\acute{\rho}\nu$. (*g*) therefor BA (Fq) is $\acute{\rho}\nu$. and g *ſch.* 12.
conſequently F is $\acute{\rho}$. *Which was to be demonſtrated.* 10.

### *Coroll.*

Hereby it appears that a rational ſuperficies may be contained under two irrational right-lines.

## PROP. CXVI. *Fig.* 11.

*Of a medial-line* AB *are produced infinite irrational-lines* BE, EF, &c. *whereof none is of the ſame kind with any of the precedent.*

Let AC be propounded $\acute{\rho}$. and AD a rectangle contained under AC, AB ; (*a*) therefore AD is $\acute{\rho}\nu$. Take BE = √ AD. (*b*) then BE is $\acute{\rho}$, and the ſame with none of the former. For no ſquare of any of the former being applied to $\acute{\rho}$, makes the breadth medial. Let the rectangle DE be finiſhed, (*a*) then DE ſhall be $\acute{\rho}\nu$, and (*b*) a *lem.* 38.
conſequently EF (√ DE) ſhall be $\acute{\rho}$, and not the ſame 10.
with any of the former, for no ſquare of the former b 11. 10.
being applied to $\acute{\rho}$, makes the latitude BE ; therefore, &c.

## PROP. CXVII. Plate V. *Fig.* 12.

*Let it be required to ſhew that in ſquare figures* BD, *the diameter* AC *is incommenſurable in length to the ſide* AB.

a 47. 1. For ACq : ABq (*a*) : : 2 : 1 (*b*) : : not Q : Q. (*c*)
b *cor.* 24. 8. therefore AC ⊔ AB. *Which was to be demonſtrated.*
c 9. 10. This Theorem was of great note with the ancient Philoſophers; ſo that he who did not underſtand it was eſteemed by *Plato* undeſerving the name of a man, but rather to be reckoned among brutes.

*The End of the Tenth* BOOK.

# The ELEVENTH BOOK
# OF
# EUCLIDE's
# ELEMENTS.

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## *Definitions.*

I. A Solid is that which hath length, breadth and thickneſs.

II. The term, or extreme of a ſolid is a Superficies.

III. A right-line AB, Plate V. *Fig.* 13. is perpendicular to a Plane CD, when it makes right-angles ABD, ABE, ABF, with all the right-lines BD, BE, BF, that touch it, and are drawn in the ſaid Plane.

IV. A Plane AB, *Fig.* 14. is perpendicular to a Plane CD, when the right-lines FG, HK, drawn in one Plane AB to the line of common ſection of the two Planes EB, and making right-angles therewith, do alſo make right-angles with the other Plane CD.

V. The inclination of a right-line AB, *Fig.* 15. to a Plane CD, is when a perpendicular AE is drawn from A the higheſt point of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular AE makes in the plane CD, to the end B of the ſaid line AB which is in the ſame plane, whereby the angle ABE which is contained under the inſiſting-line AB, and the line drawn in the plane EB is acute.

VI. The inclination of a plane AB, *Fig.* 16. to a plane CD, is an acute angle FGH contained under the right-lines FH, GH which being drawn in either of the planes AB, CD to the ſame point H of the common ſection BE, make right-angles FHB, GHB, with the common ſection BE.

VII. Planes are ſaid to be inclined to other planes in the ſame manner, when the ſaid angles of inclination are equal one to another.

VIII. Parallel planes are thoſe which being prolonged never meet.

IX. Like ſolid figures are ſuch as are contained under like planes equal in number.

X. Equal and like ſolid figures are ſuch as are contained under like planes equal both in multitude and magnitude.

XI. A ſolid angle is the inclination of more than two right-lines which touch one another, and are not in the ſame ſuperficies.

*Or thus* ;

A ſolid angle is that which is contained under more than two plane angles not being in the ſame ſuperficies, but conſiſting all at one point.

XII. A Pyramid is a ſolid figure comprehended under divers planes ſet upon one plane (which is the baſe of the pyramid,) and gathered together to one point.

XIII. A Priſm is a ſolid figure contained under planes, whereof the two oppoſite are equal, like, and parallel ; but the others are parallelograms.

XIV. A Sphere is a ſolid figure made when the diameter of a ſemicircle abiding unmoved, the ſemicircle is turned round about, till it return to the ſame place from whence it began to be moved.

*Coroll.*

Hence, all the rays drawn from the center to the ſuperficies of a ſphere, are equal amongſt themſelves.

XV. The Axis of a ſphere, is that fixed right-line, about which the ſemicircle is moved.

XVI. The Center of a ſphere, is the ſame point with the center of the ſemicircle.

XVII. The Diameter of a ſphere, is a right-line drawn thro' the center, and terminated on either ſide in the ſuperficies of the ſphere.

XVIII. A Cone is a figure made, when one ſide of a rectangled triangle (*viz.* one of thoſe that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it firſt moved. And if the fixed right-line be equal to the other which containeth the right-angle, then the Cone is a rectangled Cone: But if it be leſs, it is an obtuſe-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The Baſe of a Cone is the circle, which is deſcribed by the right-line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the ſides thereof, (namely, which contain the right-angle) abiding fix'd, till the parallelogram be turned about to the ſame place, where it began to move.

XXII. The Axis of a Cylinder is that quieſcent right-line, about which the parallelogram is turned.

XXIII. And the Baſes of a Cylinder are the circles which are deſcribed by the two oppoſite ſides in their motion.

XXIV. Like Cones and Cylinders, are thoſe both whoſe Axes and Diameters of their Baſes are proportional.

XXV. A Cube is a ſolid figure contained under ſix equal ſquares.

XXVI. A Tetraedron is a ſolid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a ſolid figure contained under eight equal and equilateral triangles.

XXVIII. A Dodecaedron is a ſolid figure contained under twelve equal, equilateral, and equiangular Pentagones.

XXIX. An Icoſaedron is a ſolid figure contained under twenty equal and equilateral triangles.

XXX. A Parallelepipedon is a ſolid figure contained under ſix quadrilateral figures, whereof thoſe which are oppoſite are parallel.

XXXI. A ſolid figure is ſaid to be inſcribed in a ſolid figure, when all the angles of the figure inſcribed are comprehended either within the angles, or in the ſides, or in the planes of the figure wherein it is inſcribed.

XXXII. Likewiſe a ſolid figure is then ſaid to be circumſcribed about a ſolid figure, when either the angles, or ſides, or planes of the circumſcribed figure touch all the angles of the figure which it contains.

### PROPOSITION I. Plate V. *Fig.* 17.

*One part* AC *of a right-line cannot be in a plane ſuperficies, and another part of it* CB *above the ſame.*

Produce AC in the plane directly to F. If you conceive CB to be drawn ſtrait from AC, then two right-
a 10. ax. 1. lines AB, AF, have one common ſegment AC. (*a*) *Which is impoſſible.*

### PROP. II. *Fig.* 18.

*If two right-lines* AB, CD, *cut one another, they are in the ſame plane: And every triangle* DEB *is in one and the ſame plane.*

For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right-line ED is in a plane, and the other
a 10. ax. 1. part elevated upwards. (*a*) *Which is abſurd.* Therefore the triangle EDB is in one and the ſame plane; and ſo alſo are the right-lines ED, EB; (*a*) wherefore the whole lines AB, DC, are in one plane. *Which was to be demonſtrated.*

### PROP. III. *Fig.* 19.

*If two planes* AB, CD, *cut one the other, their common ſection* EF *is a right-line.*

If EF the common ſection be not a right-line, (*a*)
a 1. poſt. 1. then in the plane AB draw the right-line EGF; (*a*) and in the plane CD the right-line EHF. therefore two right-
b 14. ax. 1. lines EGF, EHF include a ſuperficies. (*b*) *Which is abſurd.*

### PROP. IV. *Fig.* 20.

*If at* E *the common ſection of two right-lines* AB, CD, *a right-line* EF *ſtands at right-angles to them, it ſhall alſo be at right-angles to the plane* ACBD *drawn thro' the ſaid lines.*

Take EA, EC, EB, ED, equal one to the other, and
join the right-lines AC, CB, BD, AD draw any right-
line GH thro' E, and join FA, FC, FD, FB, FG, FH.
a conſtr. Becauſe AE is (*a*) = EB, and DE (*a*) = EC, and the
b 15. 1. angle AED (*b*) = CEB (*c*) therefore AD is = CB, (*c*)
c 4. 1. likewiſe AC = DB; (*d*) therefore AD is parallel to CB,
d ſch. 34. 1. (*d*) and AC to BD. (*e*) wherefore the angle GAE =
e 29. 1.

EBH, and the angle AGE = EHB. But alſo AE *(f)* f *conſtr.*
= EB; *(g)* therefore GE = EH, *(g)* and AG = BH. g 26. 1.
whence by reaſon of the right-angles, by the hyp. and h 4. 1.
ſo equal, at E, (*h*) the baſes FA, FC, FB, FD, are equal.
Therefore the triangles ADF, FBC, are equilateral one
to another, (*k*) and thence the angle DAF = BCF. k 8. 1.
Therefore in the triangles AGF, FBH, the ſides FG,
FH (*l*) are equal; and ſo by conſequence the triangles l 4. 1.
FEG and FEH are mutually equilateral; (*m*) therefore m 8. 1.
the angles FEG, FEH are equal, and (*n*) ſo right-angles. n 10. *def.* 1
In like manner, FE makes right-angles with all the lines
drawn thro' E in the plane AD, BC, (*o*) and is there- o 3. *def.* 11.
fore perpendicular to the ſaid plane.

## PROP. V. Plate V. *Fig.* 21.

*If a right-line* AB *be erected perpendicular to three right-lines* AC, AD, AE, *touching one the other at the common ſection, thoſe three lines are in the ſame plane.*

For AC, AD, (*a*) are in one plane FC; (*a*) and AD, a 2. 11.
AE, are in one plane BE, which if you conceive to be
ſeveral planes, then let their interſection *(b)* be the right- b 3 11.
line AG; therefore becauſe BA by the Hyp. is perpen-
dicular to the right-lines AC, AD, (*c*) and ſo to the plane c 4. 11.
FC, (*d*) it is alſo perpendicular to the right-line AG. d 3. *def.* 11
therefore (ſince (*a*) that AB is in the ſame plane with
AG, AE) the angles BAG, BAE, are right-angles, and
conſequently equal, the part and the whole. *Which is abſurd.*

## PROP. VI. *Fig.* 22.

*If two right-lines* AB, DC, *be erected perpendicular to one and the ſame plane* EF, *thoſe right-lines* AB, DC, *are parallel one to the other.*

Draw AD, whereunto let DG=AB be perpendicular
in the plane EF, and join BD, BG, AG. Becauſe in
the triangles BAD, ADG, the angles DAB, ADG (*a*) are a *hyp.*
right-angles, and AB (*b*) =DG, and AD is common, (*c*) b *conſtr.*
therefore BD = AG. whence in the triangles AGB, c 4. 1.
BGD, equilateral one to the other, the angle BAG is
(*d*) = BDG; of which ſince BAG is a right-angle, d 8. 1.
BDG ſhall be ſo alſo, but the angle GDC is ſuppoſed
right, therefore the right-line GD is perpendicular to
the three lines DA, DB, CD, (*e*) which are therefore in e 5. 1.
the ſame plane (*f*) wherein AB is. Wherefore ſince AB f 2. 1.

and CD are in the ſame plane, and the internal angles
g 28. 1. BAD, CDA, are right-angles, (*g*) AB and CD ſhall be
parallels. *Which was to be demonſtrated.*

### PROP. VII. Plate V. *Fig.* 23.

*If there are two parallel right-lines* AB, CD, *and any points* E, F, *be taken in both of them, the line* EF *which is joined at theſe points, is in the ſame plane, with the parallels* AB, CD.

Let the plane in which AB, CD, are, be cut by another plane at the points E, F; then if EF is not in the plane ABCD, it ſhall not be the common ſection. There-
a 3. 11. fore let EGF be the common ſection; which (*a*) then is
a right-line, therefore two right-lines EF, EGF, in-
b 14. *ax.* 1. clude a ſuperficies. (*b*) *Which is abſurd.*

### PROP. VIII. *Fig.* 24.

*If there are two parallel right-lines* AB, CD, *whereof one* AB, *is perpendicular to a plane* EF, *then the other* CD *ſhall be perpendicular to the ſame plane* EF.

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Book being transferr'd hither; the angles GDA, and
a 4. 11. GDB are right-angles: (*a*) Therefore GD is perpendi-
b 7. 11. cular to the plane, wherein are AD, DB, (*b*) (in which
c 3. *def.* 11. alſo AB, CD, are.) (*c*) therefore GD is perpendicular to
d 29. 1. CD; but the angle CDA is alſo (*d*) a right-angle, (*e*)
e 4. 11. therefore CD is perpendicular to the plane EF,
*Which was to be demonſtrated.*

### PROP. IX. *Fig.* 25.

*Right-lines* (AB, CD) *which are parallel to the ſame right-line* EF, *but not in the ſame plane with it, are alſo parallel one to the other.*

In the plane of the parallels AB, EF, draw HG perpendicular to EF; alſo in the plane of the parallels EF, CD, draw IG perpendicular to EF. (*a*) Therefore EG is perpendicular to the plane wherein HG, GI are; and AH, CI are perpendicular to the ſame plane, (*c*) therefore AH and CI are parallels. *Which was to be demon.*

### PROP. X. Plate V. *Fig.* 26.

*If two right-lines* AB, AC, *touching one another be parallel to two other right-lines* ED, DF, *touching one another, and not being in the ſame plane, thoſe right-lines contain equal angles,* BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AD, BC, EF, BE, CF. Since AB, DE, (*a*) are parallels and equal, (*b*) alſo BE, AD, are parallels and equal. In like manner CF, AD, are parallel and equal; (*c*) therefore alſo BE, FC, are parallel and equal. (*d*) Therefore BC, EF are equal. Wherefore ſince the triangles BAC, EDF, are of equal ſides one to the other, the angles BAC, EDF (*e*) ſhall be equal. *Which was to be demonſtrated.*

a *hyp. and conſtr.*
b 33. 1.
c 2. *ax.* 1. *and* 9. 11.
d 33. 1.
e 8. 1.

### PROP. XI. *Fig.* 27.

*From a point given on high* A *to draw a right-line* AI *perpedicular to a plane below* BC.

In the plane BC draw any line DE; to which from the point A (*a*) draw the perpendicular AF, to the ſame DE through F in the plane BC (*b*) draw the perpendicular FH, then to FH (*a*) draw the perpendicular AI, this ſhall be perpendicular to the plane BC.

For thro' I (*c*) let KIL be drawn parallel to DE. Becauſe DE (*d*) is perpendicular to AF, and FH. (*e*) therefore DE ſhall be perpendicular to the plane IFA; and ſo alſo KL (*f*) is perpendicular to the ſame plane; (*g*) therefore the angle KIA is a right-angle, but the angle AIF is alſo (*h*) a right-angle; (*l*) therefore AI is perpendicular to the plane BC. *Which was to be done.*

a 12. 1.
b 11 1.
c 31. 1.
d *conſtr.*
e 4. 11.
f 8. 11.
g 3. *def.* 11.
h *conſtr.*
l 4. 11.

### PROP. XII. *Fig.* 28.

*In a plane given* BC, *at a point given therein* A, *to erect a perpendicular line* AF.

From ſome point D without the plane, (*a*) draw DE perpendicular to the ſaid plane BC, and joining the points A, E, by a line AE, (*b*) draw AF parallel to DE (*c*) it is apparent that AF is perpendicular to the plane BC. *Which was to be done.*

a 11. 11.
b 31. 1.
c 8. 11.

This and the preceding problem are practically performed by applying two Squares to the point given; as appears by 4. 11.

PROP.

## PROP. XIII. Plate V. *Fig* 29.

*At a point given* C *in a plane given* AB, *two right-lines* CD, CE, *cannot be erected perpendicular to the ſaid given plane on the ſame ſide.*

a 6. 11. For both CD, and CE (*a*) ſhould then be perpendicular to the plane AB, and conſequently parallel; which is repugnant to the definition of parallel-lines.

## PROP. XIV. *Fig.* 30.

*Planes* CD, FE, *to which the ſame right-line* AB *is perpendicular, are parallel.*

If you deny this; then let the planes CD, EF meet, ſo that their common ſection be the right-line GH, in which take any point I, draw to it the right-lines IA, IB, in the ſaid planes, whereby in the triangle IAB, two
*hyp.* and a *def.* 11. angles IAB, IBA (*a*) are right-angles. (*b*) *Which is abſurd.*
b 17. 1.

## PROP. XV. *Fig.* 31.

*If two right-lines* AB, AC, *touching one the other, are parallel to two other right-lines* DE, DF, *touching one the other, and not being in the ſame plane with them, the planes* BAC, EDF, *drawn by thoſe right-lines are parallel one to the other.*

a 11. 11. From A (*a*) draw AG perpendicular to the plane EF.
b 31. 1. (*b*) and let GH, GI be parallel to DE, DF. (*c*) theſe alſo
c 9. 11. ſhall be parallel to AB, AC. Therefore ſince the an-
d 3. *def.* 11. gles, IGA, HGA, (*d*) are right-angles, alſo CAG, BAG,
e 29. 1. (*e*) ſhall be right-angles; (*f*) therefore GA is perpendi-
f 4. 11. cular to the plane BC; but the ſame is perpendicular
g *conſtr.* to the plane EF, (*h*) therefore the planes BC, EF, are
h 14. 11. parallel. *Which was to be demonſtrated.*

## PROP. XVI. *Fig.* 32.

*If two parallel planes* AB, CD, *are cut by ſome other plane* HEIGF, *their common ſections* EH, GF *are parallel one to the other.*

For if they are ſaid to be not parallel, then, ſince they are in the ſame cutting plane, they muſt meet ſome where, ſuppoſe in I, wherefore ſince the whole lines
a 2. 11. HEI, FGI (*a*) are in the planes AB, CD, produced, the planes alſo ſhall meet. *contrary to the Hyp.*

## PROP. XVII. Plate V. *Fig.* 33.

*If two right-lines* ALB, CMD, *are cut by parallel planes* EF, GH, IK: *they ſhall be cut proportionally,* (AL : LB : : CM : MD.)

Let the right-lines AC, BD, be drawn in the planes EF,
IK; as alſo AD meeting the plane GH in the point N,
and join NL, NM, the planes of the triangles ADC,
ADB, make the ſections BD, LN, and AC, NM, (*a*) a 16. 11.
parallels. Therefore AL : LB : : AN : ND (*b*) : : b 2. 6.
CM : MD. *Which was to be demonſtrated.*

## PROP. XVIII. *Fig.* 34.

*If a right-line* AB *be perpendicular to ſome plane* CD, *all the planes* EF *paſſing thro' that right-line* AB *ſhall be perpendicular to the ſame plane* CD.

Let there be ſome plane BF drawn thro' AB, making
the ſection EG with the plane CD; from ſome point
whereof H, (*a*) draw HI parallel to AB in the plane a 31. 1.
EF; (*b*) then ſhall HI be perpendicular to the plane CD, b 8. 11.
and ſo likewiſe any other lines, that are perpendicu-
lar to EG; (*c*) therefore the plane EF is perpendicular c 4. *def.* 11.
to the plane CD; and for the ſame reaſon any other
planes drawn thro' AB ſhall be perpendicular to CD.
*Which was to be demonſtrated.*

## PROP. XIX. *Fig.* 35.

*If two planes* AB, CD, *cutting one the other, are perpendicular to ſome plane* GH, *their line of common ſection* EF *ſhall be perpendicular to the ſame plane* (GH.)

Becauſe the planes AB, CD, are taken perpendicular
to the plane GH, it appears by 4. def. 11. that from the
point F there may be drawn in both planes AB, CD, a
perpendicular to the plane GH, which ſhall be (*a*) one a 13. 11.
and the ſame line, and therefore the common ſection of
the ſaid planes. *Which was to be demonſtrated.*

## PROP. XX. *Fig.* 36.

*If a ſolid angle* ABCD *be contained under three plane angles,* BAD, DAC, BAC, *any two of them howſoever taken are greater than the third.*

If the three angles are equal, the aſſertion is evident;
if unequal, then let the greateſt be BAC; from whence (*a*) a 23. 1.

take away BAE = BAD; and make AD = AE; and alſo draw BEC, BD, DC.

Becauſe the ſide BA is common, and AD (*b*) = AE; b *conſtr.*
c 4. 1. and the angle BAE (*b*) = BAD, (*c*) thence is BE = BD.
d 20. 1. but BD + DC is (*d*) ⊏ BC; therefore DC ⊏ EC.
e 5. *ax.* 1. Wherefore ſince AD (*b*) = AE, and the ſide AC is com-
f 25. 1. mon, and DC ⊏ EC, (*f*) the angle CAD ſhall be ⊏
g 4. *ax.* 1. EAC, (*g*) therefore the angle BAD + CAD ⊏ BAC.
*Which was to be demonſtrated.*

## PROP. XXI. Plate V. *Fig.* 37.

*Every ſolid angle* A *is contained under leſs angles than four plane right-angels.*

For let a plane any wiſe cutting the ſides of the ſolid angle A make a many-ſided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X; and I term the ſum of the angels at the baſes of the triangle Y;
a 32. 1. *and* whereof X + 4 right-angles (*a*) = Y + A, but becauſe
*ſch.* 31. 1. that (of all the angles at B) (*b*) the angle ABE + ABC
b 20. 11. is ⊏ CBE, and the ſame is true alſo of the angles
c 5. *ax.* 1. at C, at D, and at E, (*c*) it is manifeſt that Y is
⊏ X, and conſequently A ſhall be ⊐ 4 right-angles.
*Which was to be demonſtrated.*

## PROP. XXII. *Fig.* 38.

*If there are three plane angles* A, B, HCI, *whereof two howſoever taken are greater than the third, and the right-lines which contain them are equal* AD, AE, FB, &c. *then of the right-lines* DE, FG, HI, *connecting thoſe equal right-lines together, it is poſſible to make a triangle.*

a 22. 1. A triangle may be (*a*) made of them, if any two
b 23. 1. be greater than the third; but they are ſo. For (*b*)
make the angle HCK = B, and CK = CH, and draw
c 4. 1. HK, IK; (*c*) thence KH = FG, and becauſe the angle
d *hyp.* KCI (*d*) ⊏ A; (*e*) therefore KI ⊏ DE, but KI (*f*)
e 24. 1. ⊐ HI + KH (FG;) therefore DE ⊐ HI + FG.
f 20. 1. By the like argument any other two may be proved
greater than the third; and conſequently (*a*) it is
poſſible to make a triangle of them. *Which was to be demonſtrated.*

### PROP. XXIII. Plate V. *Fig.* 39.

*To make a ſolid angle* MHIK *of three plane angles*
A, B, C, *whereof two howſoever taken are greater than the*
*third.* * *But it is neceſſary that thoſe three angles be leſs* * 21. 11.
*than four right-angles.*

Make AD, AE, BE, BF, CF, CG, equal one
to the other; and of the ſubtended-lines DE, EF,
FG (that is, of the equal lines GH, IK, KH) (*a*) make a 22. 11.
the triangle HKI; about which (*b*) deſcribe the circle *and* 21. 1.
LHKI. * But becauſe AD is ⊏ HL, (*c*) let ADq be = b 5. 4.
HLq + LMq. (*d*) and let LM be perpendicular to the * *See Cla-*
plane of the circle HKI; and draw HM, KM, IM. *vius*
Wherefore ſince the angle HLM (*e*) is a right-angle. (*f*) c *ſch.* 47. 1.
thence is MHq = HLq + LMq (*g*) = ADq. therefore d 12. 11.
MH = AD. By the ſame way of reaſoning MK, MI, e 3. *def.* 11.
AD (that is AE, AB, &c.) are equal; therefore ſince f 47. 1.
HM = AD, and MI = AE, and DE (*h*) = HI, (*k*) the g *conſtr.*
angle A ſhall be = HMI, (*k*) as likewiſe the angle IMK h *conſtr.*
= B, (*k*) and the angle HMK = C, wherefore a ſolid k 8. 1.
angle is made at M of the three given plane angles.
*Which was to be done.* AD is aſſumed to be ⊏ HL.
But this is manifeſt. For if AD be = or ⊐ HL, then
is the angle A (*l*) = (*m*) or ⊏ HLI. In like manner l *conſtr.* &
ſhall B be equal or ⊏ HLK, and C = or ⊏ KLI. 8. 1.
wherefore A + B + C * ſhall either equal or exceed m 21. 1.
four right-angles, *contrary to the Hyp.* therefore rather *4 *cor.* 13. 1.
let AD be ⊏ HL. *Which was to be demonſtrated.*

### PROP. XXIV. *Fig.* 40.

*If a ſolid* AB *be contained under parallel planes, the oppoſite planes thereof* (AG, BD, &c.) *are like and equal parallelograms.*

The plane AC cutting the parallel planes AG, DB, (*a*) a 16. 11.
makes the ſections AH, DC, parallels, and for the ſame
reaſon AD, HC are parallels. Therefore ADCH is a
pgr. By the like argument the other planes of the parallelepipedon are (*b*) pgrs. wherefore ſince AF is paral- b 35. *def.* 1.
lel to HG, and AD to HC, (*c*) the angle FAD ſhall be c 10. 11.
= GHC, therefore becauſe AF (*d*) = HG, and AD (*d*) = d 34. 1.
HC, and ſo AF : AD : : HG : HC, the triangles FAD, e 7. 5.
GHC (*g*) are like and (*h*) equal; and conſequently pgrs. g 6. 6.
AE, HB are like and (*k*) equal, and the ſame may be h 4. 1.
ſhewn of the reſt of the oppoſite planes, therefore, &c. k 6. *ax.* 1.

PROP.

## P R O P. XXV. Plate V. *Fig.* 41.

*If a ſolid Parallelepipedon* ABCD *be cut by a plane* EF *parallel to the oppoſite planes* AD, BC ; *then as the baſe* AH *is to the baſe* BH, *ſo ſhall ſolid* AHD *be to ſolid* BHC.

Conceive the parallelepipedon ABCD to be extended on either ſide, and take AI = AE, and BK = EB; and put the planes IQ, KP, parallel to the planes AD,
a 36. 1. *and* BC; then the pgrs. IM, AH, and (*a*) DL, DG, (*b*) and
1 *def.* 6. IQ, AD, EF, &c. are (*a*) like and equal (*c*) wherefore
b 24. 11. the parallelepipedon AQ is = AF ; and for the ſame
c 10. *def.* reaſon the parallelepipedon BP = BF ; therefore the ſo-
11. lids IF, EP are as multiple of the ſolids AF, EC, as the baſes IH, KH, are of the baſes AH, BH. And if
d 24. 11. the baſe IH be $\sqsubset$, =, $\sqsupset$ KH, (*d*) likewiſe ſhall the
*and* 9. *def.* ſolid IE be $\sqsubset$, =, $\sqsupset$ EP ; (*e*) conſequently AH : BH
11. : : AF : EC. *Which was to be demonſtrated.*

e 6. *def.* 5. *The ſame may be accommodated to all ſorts of priſmes; whence.*

### *Coroll.*

If any priſm whatſoever be cut by a plane parallel to the oppoſite planes, the ſection ſhall be a figure equal and like to the oppoſite planes.

## P R O P. XXVI. *Fig.* 42.

*Upon a right-line given* AB, *and at a point given in it* A, *to make a ſolid angle* AHIL *equal to a ſolid angle given* CDEF.

a 11. 11. From ſome point F (*a*) draw FG perpendicular to the plane DCE, and draw the right-lines DF, FE, EG, GD, CG. Make AH = CD, and the angle HAI = DCE, and AI = CE ; and in the plane HAI make the angle HAK = DCG, and AK = CG, then erect KL perpendicular to the plane HAI, and let KL be = GF, and draw AL. Then AHIL ſhall be a ſolid angle equal to that given CDEF. For the conſtruction of this does wholly reſemble the framing of that, as will eaſily appear to any who examine it.

## PROP. XXVII. Plate V. *Fig.* 43.

*Upon a right-line given* AB *to deſcribe a parallelepipedon* AK, *like, and in like manner ſituate, with a ſolid parallelepipedon given* CD.

Of the plane angles, BAH, HAI, BAI, which are
equal to FCE, ECG, FCG, (*a* make the ſolid angle A a 26. 11.
equal to the ſolid angle C. Alſo (*b*) make FC : CE : : b 12. 6.
BA : AH (*b*) and CE : CG : : AH : AI (whence by c 22. 5.
equality (*c*) FC : CG : : BA : AI) and finiſh the parallelepipedon AK, which ſhall be like to that which is given.

For by the conſtruction, the Pgr. (*d*) BH is like to FE, d 1. *def.* 6.
and (*d*) HI to EG, and (*d*) BI to FG, and (*e*) ſo the e 24. 11.
oppoſites of theſe to the oppoſites of them : Therefore the ſix planes of the ſolid AK are like to the ſix
planes of the ſolid CD; (*f*) and conſequently AK,
CD, are like ſolids. *Which was to be demonſtrated.* f 9 *def.* 11.

## PROP. XXVIII. *Fig.* 44.

*If a ſolid parallelepipedon* AB *be cut by a plane* FGCD *drawn thro' the diagonal-lines* DF, CG, *of the oppoſite planes* AE, HB, *that ſolid* AB *ſhall be equally biſected by the plane* FGCD.

For becauſe DC, FG, are (*a*) equal and parallels, (*b*) a 24. 1.
the plane FGCD is a Pgr. and becauſe (*a*) the Pgrs. AE b 34. 1.
HB, are equal and like, (*b*) alſo the triangles AFD, HGC, CGB, DFE are equal and like. But the Pgrs. AC, AG, are equal and like to FB and FD; therefore all the planes of the priſm FGCDAH are equal and like to all the planes of the priſm FGCDEB,
and (*c*) conſequently this priſm is equal to that. c 9. *def.* 11.
*Which was to be demonſtrated.*

## PROP. XXIX. *Fig.* 45.

*Solid parallelepipedons* AGHEFBCD, AGHEMLKI, *being conſtituted upon the ſame baſe* AGHE, *and* * *in the ſame height, whoſe inſiſting lines* AF, AM, *are placed in the ſame right-lines* AG, FL, *are equal one to the other.*

* *i. e. between the parallel planes* AGHE, FLKD, *& ſo underſtand it in the fol-*

For

a 10. *def* 11 For (*a*) if from the equal priſmes AFMEDI, GBLH-
& 35. 1. CK, the common priſm NBMPCI be taken away, and
b 3. *and* 2. the ſolid AGNEHP be added, the Parallelepipedon
*ax.* 1. AGHEFBCD ſhall be = AGHEMLKI. *Which was to be demonſtrated.*

## PROP. XXX. Plate V. *Fig.* 46.

*Solid parallelepipedons* ADBCHEFG, ADCBIMLK *being conſtituted upon the ſame baſe* ADBC, *and in the ſame height, whoſe inſiſting lines* AH, AI *are not placed in the ſame right-lines, are equal one to the other.*

For produce the right-lines HEO, GFN, and LMO,
a 34. 1. KIP; and draw AP, DO, BQ, CN; (*a*) then ſhall DC,
AB, HG, EF, PQ, ON be as well equal and parallel
b 29. 11. one to the other as AD, HE, GF, BC, KL, IM, QN,
PO; (*b*) wherefore the parallelepipedon ADCBPONQ
c 1. *ax.* 1. ſhall be equal to either parallelepipedon ADCBHEFG,
ADCBIMLK; and (*c*) conſequently theſe two are equal one to the other. *Which was to be dem.*

## PROP. XXXI. Plate VI. *Fig.* 1.

* *by height underſtand the perpendicular drawn from the plane of the baſe to the oppoſite plane.*

*Solid parallelepipedons,* ALEKGMBI, CPOHQDN, *being conſtituted upon equal baſes* ALEK, CPωO, *and* * *in the ſame height are equal, one to the other.*

First, let the parallelepipedons AB, CD, have the ſides perpendicular to the baſes, and at the ſide CP
a 18. 6. being produced, (*a*) make the Pgr. PRTS equal and like to the pgr. KELA. (*b*) and ſo the parallelepipedon
b 27. 11 & 10. *def.* 11. PRTSQVYX equal and like to the parallelepipedon AB.
Produce OωE, NDδ, ωPZ, DQF, ERB, δVγ, TSZ,
YXF; and draw Eδ, Bγ, ZF.
c 30. *def.* 11 The planes Oεδ N, CRVH, ZTYF, (*c*) are parallels
d *hyp.* and 35. 1. one to the other; (*d*) and the Pgrs. ALEK, CPωO-
PRTS, PRBZ are equal. Therefore ſince the paralle-
e 25. 11. lepipedon CD : PVδω (*e*) : : Pgr. Cω (PRBZ) : Rω
f 9. 5. : : parallelepipedon PRBZQVγF : PVδω; the pa-
g 29. 11. rallelepipedon CD (*f*) ſhall be = PRBZQVγF (*g*) =
h *conſtr.* PRVQSTYX (*h*) = AB. *Which was to bo dem.*

But if the parallepipedons AB, CD, have ſides oblique to the baſe, then on the ſame baſes and in the ſame height place parallelepipedons whoſe ſides are per-

pen-

Plate V. *Facing pag. 208.*

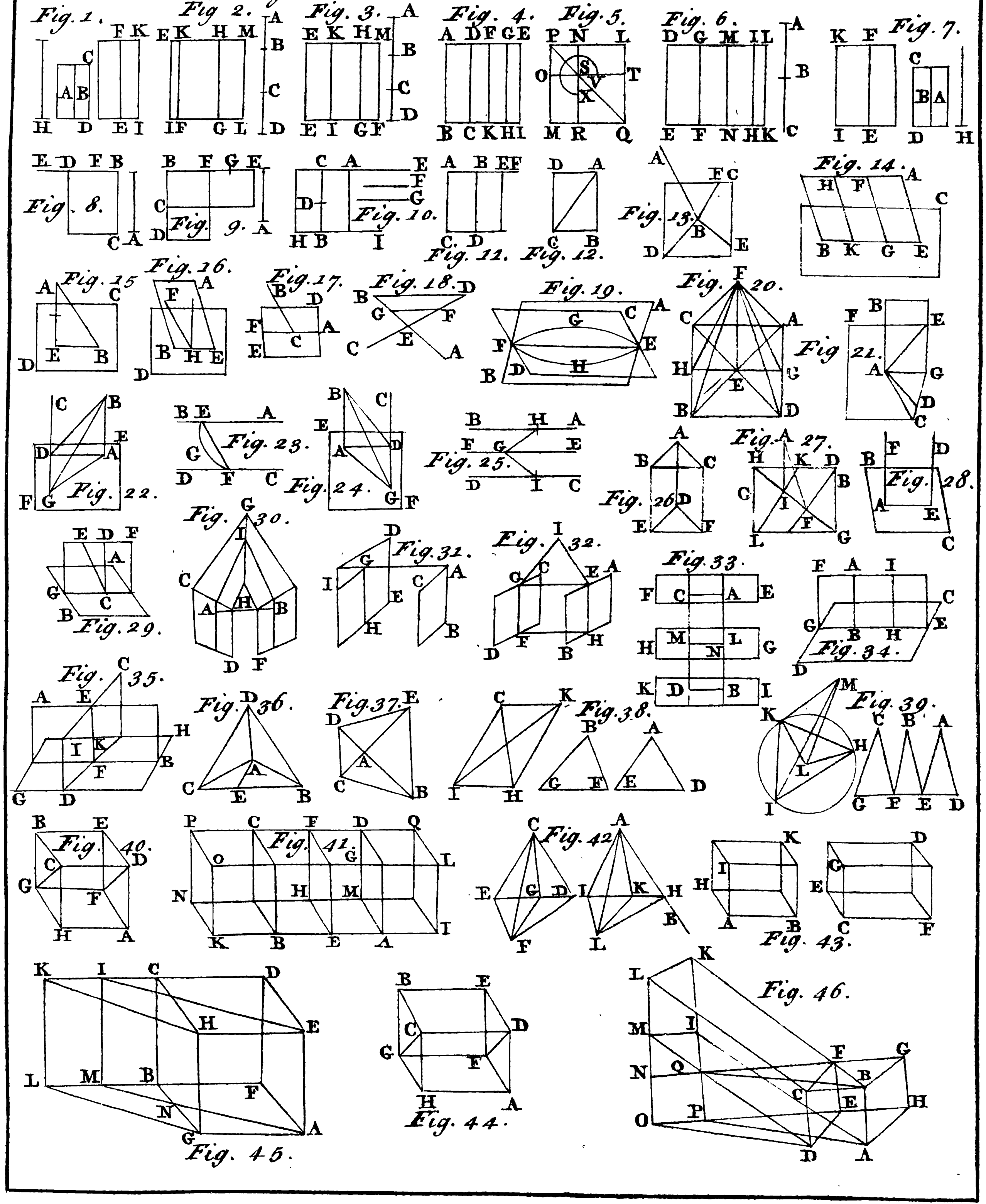
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14.
Fig. 15
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26
Fig. 27.
Fig. 28.
Fig. 29.
Fig. 30.
Fig. 31.
Fig. 32.
Fig. 33.
Fig. 34.
Fig. 35.
Fig. 36.
Fig. 37.
Fig. 38.
Fig. 39.
Fig. 40.
Fig. 41.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46.

pendicular to the baſe. (*k*) They ſhall be equal to one k 29. 11.
another, and to thoſe that are oblique, (*m*) whence alſo m 1. *ax.* 1.
the oblique parallelepipedons AB, CD are equal. *Which was to be demonſtrated.*

### PROP. XXXII. Plate VI. *Fig.* 2.

*Solid parallelepipedons* ABCD, EFGL, *of the ſame height, are one to the other, as their baſes,* AB, EF.

Produce EHI, (*a*) and make the pgr. FI ≐ AB, and a 45. 1.
(*b*) compleat the parallelepipedon FINM. It is clear that b 27. 11.
the parallelepipedon FINM: (*c*) (ABCD) EFGL (*d*) : : c 31. 11.
FI (AB) : EF. *Which was to be demonſtrated.* d 25. 11.

### PROP. XXXIII. *Fig.* 3.

*Like ſolid parallelepipedons,* ABCD, EFGH, *are to one another in triplicate ratio of their homologous ſides* AI, EK.

Produce the right-lines AIL, DIO, BIN, and (*a*) make a 3. 1.
IL, IO, IN, equal to EK, KH, KF, (*b*) and ſo the parallelepipedon b 27. 11.
IXMT equal and like to the parallelepipedon
EFGH. (*c*) Let the parallepps. IXPB, DLYQ c 31. 1.
be finiſhed. (*d*) Then ſhall be AL : IL (EK) : : DI : IO d *hyp.*
(HK) : : BI : IN (KF) (*e*) ; that is the pgrs. AD : DL e 1. 6.
: : DL : IX : : BO : IT. (*f*) *i. e.* the parallepp. ABCD f 32. 11.
: DLQY : : DLQY : IXBP : : IXBP : IXMT. (*g*) g *conſtr.*
(EFGH) (*h*) therefore the proportion of ABCD to EFGH h 10. *def.* 5.
is triplicate of the proportion of ABCD to DLQY, k 1. 6.
(*k*) or of AI to EK. *Which was to be demonſtrated.*

#### *Coroll.*

Hence it appears that if four right-lines be continually proportional, as the firſt is to the fourth, ſo is a parallelepipedon deſcribed on the firſt to a parallelepipedon deſcribed on the ſecond, being like and in like manner deſcribed.

### PROP. XXXIV. *Fig.* 4.

*In equal ſolid parallelepipedons* ADCB, EHGF, *the baſes and altitudes are reciprocal* (AD : EH : : EG. AC) *And ſolid parallelepipedons,* ADCB, EHGF, *whoſe baſes and altitudes are reciprocal, are equal.*

Firſt, let the ſides CA, GE be perpendicular to the baſes; then if the altitudes of the ſolids are equal, the baſes alſo ſhall be equal, and the thing is clear. But if

a 3. 1.
b 31. 1.
c 32. 11.
d 17. 5.
e 1. 6.
f *conſtr.*
g 11. 5.
h 32. 11.
k *hyp.*
l 1. 6.
m 32. 11.
n 9. 5.

the altitudes are unequal, from the greater EG (*a*) take EI = AC, and at I (*b*) draw the plane IK parallel to the baſe EH. Then

1. *Hyp.* AD : EH (*c*) : : parallepp. ADCB : EHIK (*d*) : : parallepp. EHGF : EHIK (*c*) : : GL : IL (*e*) : : GE : IE (*f*) (AC) ; it is plain therefore (*g*) that AD : EH : : GE : AC. *Which was to be demonſtrated.*

2. *Hyp.* ADCB : EHIK (*h*) : : AD : EH (*k*) : : EG : EI (*l*) : : GL : IL (*m*) : : parallepp. EHGF : EHIK ; (*n*) wherefore the parallelepipedon ADCB = EHGF. *Which was to be demonſtrated.*

Moreover, let the ſides be oblique to the baſes, and erect right parallelepipedons upon the ſame baſes with the ſame altitude ; the oblique parallepps. ſhall be equal to them. Wherefore ſince by the firſt part, the baſes and altitudes of thoſe are reciprocal, the baſes and altitudes of theſe alſo ſhall be reciprocal. *Which was to be demon.*

### *Coroll.*

*All that hath been dem. of parallelpps. in the* 29, 30, 31, 32, 33, 34. Prop. *does alſo agree to triangular priſms, which are half parallelpps. as appears by Prop.* 28. Therefore,

1. Triangular priſms are of equal height with their baſes.
2. If they have the ſame or equal baſes and the ſame altitude, they are equal.
3. If they are like, their proportion is triplicate of that of their homologous ſides.
4. If they are equal, their baſes and altitudes are reciprocal ; and if their baſes and altitudes are reciprocal, they are alſo equal.

### P R O P. XXXV. Plate VI. *Fig.* 5.

*If there are two plane angles* BAC, EDF, *equal, and from the point of thoſe angles two righ-lines* AG, DH, *be elevated on high, containing equal angles with the lines firſt given, each to his correſpondent angle* (*the angle* GAB = HDE, *and* GAC = HDF) *and if in thoſe elevated lines* AG, DH, *ſome points be taken,* G, H ; *and from theſe points perpendicular lines* GI, HK, *drawn to the planes* BAC, EDF, *in which the angles firſt given are, and right-lines* AI, DK, *be drawn to the angles firſt given from the points* I, K, *which are made by the perpendiculars in the planes ; thoſe right-lines with the elevated lines* AG, DH, *ſhall contain equal angles* GAM, HDK.

Make DH, AL, equal; and GI, LM parallels, and
MC to AC, MB to AB, KF to DF, KE to DE perpendicular;
and draw the right-lines BC, LB, LC, and EF,
HF, HE; (*a*) and LM perpendicular to the plane a 8. 11.
BAC; (*b*) wherefore the angles LMC, LMA, LMB; b 3. *def.* 11.
and for the ſame reaſon the angles HKF, HKD, HKE,
are right-angles. Therefore ALq (*c*) = LMq + AMq c 47. 1.
(*c*) = LMq + CMq + ACq (*c*) = LCq + ACq; (*d*) d 48. 1.
therefore the angle ACL is a right-angle. Again ALq (*e*) e 47. 1.
= LMq + MAq (*e*) = LMq + BMq + BAq (*e*) = BLq
+ BAq; (*d*) therefore the angle ABL is alſo a right-angle.
By the like inference the angles DFH, DEH
are right-angles; *(f)* therefore AB = DE, *(f)* and BL f 26. 1.
= EH, *(f)* and AC = DF, and CL = FH; (*g*) g 4. 1.
wherefore alſo BC = EF; (*g*) and the angle ABC =
DEF, (*g*) and the angle ACB = DFE; (*h*) whence the h 3. *ax.* 1.
other right-angles CBM, BCM, are equal to the other k 26. 1.
FEK, EFK (*k*) therefore CM = FK, (*l*) and ſo alſo l 47. 1.
AM = DK; therefore if from LAq (*m*) = HDq be m *conſtr.*
taken away AMq = DKq, (*n*) there remains LMq = n 47. 1. &
HKq; wherefore the triangles LAM, HDK are equilateral 3. *ax.*
one to the other; (*p*) therefore the angle LAM p 8. 1.
= HDK. *Which was to be demonſtrated.*

*Coroll.*

Therefore, if there be two plane angles equal, from whoſe points equal right-lines are elevated on high containing equal angles with the lines firſt given, each to each; perpendiculars drawn from the extreme points of thoſe elevated lines to the planes of the angles firſt given, are equal one to the other, *viz.* LM = HK.

## PROP. XXXVI. Plate VI. *Fig.* 6.

*If there are three right-lines* DE, DG, DF *proportional, the ſolid parallelpp.* DH. *made of them, is equal to the ſolid parallelpp.* IN *made of the middle-line* DG (IL) *which is alſo equilateral, and equiangular to the ſaid parallelepipedon* DH.

Becauſe DE : IK (*a*) : : IL : DF, (*b*) the pgr. LK a *hyp.*
ſhall be = FE, and by reaſon of the equality of the b 14 6.
plane angles at D and I, and of the lines GD, IM,
alſo the altitudes of the parallelpps. are equal by the
preceding Coroll. (*c*) therefore the parallelps. are equal c 31. 11.
one to the other. *Which was to be demonſtrated.*

## PROP. XXXVII. Plate VI. *Fig.* 7.

*If there are four right-lines* A, B, C, D, *proportional, the ſolid parallelepps.* A, B, C, D *being like, and in like ſort deſcribed from them, ſhall be proportional. And if the ſolid parallelpps. being like and in like ſort deſcribed, be proportional* (A: B : : C : D.) *then thoſe right-lines* A, B, C, D, *ſhall be proportional.*

a 33. 11. For the proportions of the parallelepps. (*a*) are triplicate
b *ſch.* 23. 5. of thoſe lines; therefore if A : B : : C : D, (*b*) then
ſhall the parallelpp. A : parallelpp. B : : parallelpp. C :
parallelp. D. and ſo alſo contrarily.

## PROP. XXXVIII. *Fig.* 8.

*If a plane* AB *be perpendicular to a plane* AC, *and a perpendicular-line* EF *be drawn from a plane* E *in one of the planes* (AB) *to the other point* AC, *that perpendicular* EF *ſhall fall upon the common ſection of the planes* AD.

If it be poſſible, let F fall without the interſection
a 12. 1. AD, and in the plane AC, (*a*) draw FG perpendicular
b 4. *and* 3. to AD, and join EG. The angle FGE (*b*) is a right-angle,
*def.* 11 and EFG is ſuppoſed to be ſuch alſo; therefore two
c 17. 1. right-angels are in the triangle EFG. (*c*) *Which is abſurd.*

## PROP. XXXIX. *Fig.* 9.

*If the ſides* (AE, FC, AF, EC, *and* DH, GB, DG, HB) *of the oppoſite planes* AC, DB, *of a ſolid parallelepipedon* AB, *be divided into two equal parts, and planes* ILQO, PKMR, *be drawn thro' their ſections, the common ſection of the planes* ST, *and the diameter of the ſolid parallelepipedon* AB *ſhall divide one the other into two equal parts.*

a 34. 1. Draw the right-lines SA, SC. TD, TB. Becauſe (*a*)
b 29. 1. the ſides DO, OT are equal to the ſides BQ, QT, (*b*)
c 4. 1. and the alternate-angles TOD, TQB equal, alſo (*c*) the
d *ſch.* 15. 1. baſes DT, TB, and the angels DTO, BTQ are equal, (*d*)
e 34. 1. therefore DTB is a right-line, and ſo in like manner is
f 9. 11. *and* ASC. Moreover (*e*) as well AD is parallel and equal to
1 *ax.* FG (*e*) as FG to CB. and (*f*) thence AD is parallel and
g 33. 1. equal to CB; (*g*) and conſequently AC to DB (*h*) wherefore
h 7. 1. 11 AB and ST are in the ſame plane ABCD. Therefore
ſince the vertical-angles AVS, BVT, and the alternate-

nate-angles ASV, BTV are equal (*k*) and AS = BT; k 7. *ax.* 1.
therefore ſhall AV be = BV, (*l*) and SV = VT. *Which* l 26. 1.
*was to be dem.*

### *Coroll.*

Hence in every parallelepipedon, all the diameters biſect one another in one point, V.

## PROP. XL. Plate VI. *Fig.* 10.

*If two priſms* ABCFED, GHMLIK, *be of equal altitude, whereof one hath its baſe* ABCF *a parallelogram, and the other* GHM *a triangle, and if the parallelogram* ABCF *be double to the triangle* GHM; *thoſe priſms* ABCFED, GHMLIK *are equal.*

For if the parallelepipedons AN, GQ, be compleated, a 31. 11.
(*a*) they ſhall be equal, becauſe of the equality (*b*) of the b 34. 1. & 7. *ax.*
baſes AC, GP, and (*c*) of the altitudes; (*d*) therefore alſo c *hyp.*
the priſms, (*e*) the halfs thereof ſhall be equal. *Which* d 28. 11.
*was to be dem.* e 7. *ax.* 1.

### *Schol.*

*From the preceding demonſtrations, the demenſion of triangular priſms, and quadrangular, or parallelepipedons, is learnt*; viz. *by multiplying the altitude into the baſe.* Andr. Tacq.

As if the altitude be 10 foot, and the baſe 100 ſquare foot (the baſe may be meaſured by *ſch.* 35. 1. or by 41. 1.) then multiply 100 by 10, and 1000 cubic foot ſhall be produced for the ſolidity of the priſm given.

For as a rectangle, ſo alſo is a right parallelepp. produced from the altitude multiplied into the baſe. Therefore every parallelepipedon is produced from the altitude multiplied into the baſe, as appears by 31. of this Book.

Moreover, ſince the whole parallelepipedon is produced from the altitude drawn into the baſe, the half thereof (that is, a triangular priſm) ſhall be produced from the altitude drawn into half the baſe, namely the triangle.

### *An Advertiſement.*

Obſ. *That of thoſe letters which denote a ſolid angle, the firſt is always at the point in which the angle is; but of thoſe letters which denote a pyramid, the laſt is at the ſupreme point thereof.*

Ex. gr. the ſolid angle ABCD is at the point A; and the ſupreme point of the pyramid BCDA is at the point A, and the baſe is the triangle BCD.

## *The End of the Eleventh* BOOK.

# The TWELFTH BOOK OF EUCLIDE's ELEMENTS.

## PROPOSITION I. Plate VI. *Fig.* 11, 12.

*LIKE polygonous figures* ABCDE, FGHIK *inſcribed in circles* ABD, FGI, *are one to another, as the ſquares deſcribed on the diameters of the circles* AL, FM.

Draw AC, BL, FH, GM. Becauſe (*a*) the angle a 1. *def.* 6.
ABC = FGH, (*a*) and AB : BC : : FG : GH (*b*) there- b 6. 6.
fore ſhall the angle ACB (*c*) (ALB) be = FHG (*c*) (FMG.) c 21. 3.
but the angles ABL, FGM (*d*) are right and ſo equal; d 31. 3.
(*e*) therefore the triangles ABL, FGM are equiangular. (*f*) e 32. 3.
wherefore AB : FG : : AL : FM. (*g*) therefore ABCDE : f *cor.* 4. 6.
FGHIK : : ALq : FMq. g 22. 6.

### *Coroll.*

Hence (becauſe AB : FG : : AL : FM : : BC : GH
&c,) the ambits of like polygonous figures inſcribed in a
circle are in the ſame (*h*) proportion as the diameters. h 1. 12. & 12. 5.

## PROP. II. *Fig.* 13, 14.

*Circles* ABT, EFN, *are in proportion one to another, as the ſquares of their diameters* AC, EG *are.*

Suppoſe ACq : FGq : : the circle ABT : I. I ſay then I is equal to the circle EFN.

For firſt, if it be poſſible, let I be leſs than the circle
EFN, and let K be the exceſs or difference. Inſcribe
the ſquare EFGH in the circle EFN, (*a*) it being the a *ſch.* 7. 4.
half of a circumſcribed ſquare, and ſo greater than
the ſemicircle. (*b*) Divide equally in two the arches b 30. 3.

EF, FG, GH, HE, and at the points of the diviſions
join the right-lines EL, LF, *&c.* thro' L draw the
c *ſch.* 27. 3. tangent PQ (*c*) which is parallel to EF; and produce
d 41. 1. HEP, GFQ, then is the triangle ELF (*d*) the half of the
pgr. EPQF, and ſo greater than the half of the ſegment
ELF; and in like mnaner the reſt of thoſe triangles
exceed the halfs of the reſt of the ſegments. And if
the arches E L, L F, F M, *&c.* be again biſected,
and the right-lines joined, the triangles will likewiſe
exceed the half of the ſegments. Wherefore if the
ſquare EFGH be taken from the Circle EFN, and the
triangles from the other ſegments, and this to be done
e 1. 10. continually, at length (*e*) there will remain ſome
magnitude leſs than K. Let us have gone ſo far, namely,
to the ſegments EL, LF, FM, *&c.* taken together
f *hyp. and* leſs than K. Therefore I (*f*) (the circle EFN — K) ⊐
3. *ax.* the polyg. ELFMGNHO (the circle EFN — the ſegment
g 30. 3. & EL + LF, *&c.*) In the circle ABT (*g*) conceive
1. *poſt.* 1. a like polygon AKBSCTDV inſcribed. Therefore ſince
h 1. 12. AKBSCTDV : ELFMGNHO (*h*) : : ACq : EGq (*k*) : :
k *hyp.* the circle ABT : I. and the polyg. AKBSCTDV (*l*) ⊐
l 9. *ax.* 1. the circle ABT. the polyg. ELFMGNHO (*m*) ſhall be
m 14. 5. ⊐ I. but before, I was ⊐ ELFMGNHO, which is
repugnant.

Again, if it be poſſible, let I be ⊏ the circle EFN.
n *hyp.* Therefore becauſe ACq : EGq (*n*) : : the circle ABT : I;
and inverſely I : the circle ABT : : EGq : HCq. ſuppoſe
o 14. 5. I : the circle ABT : : the circle EFN : K. (*o*) therefore
p 11. 5. the circle ABT ⊏ K. (*p*) and EGq : ACq : : the circle
EFN : K, which was juſt now ſhewn to be repugnant.

Therefore it muſt be concluded, that I is = to the circle EFN. *Which was to be demonſtrated.*

*Coroll.*

Hence it follows, that as a circle is to a circle, ſo is a polygon inſcribed in the firſt to a like polygon inſcribed in the ſecond.

## PROP. III. Plate VI. *Fig.* 15.

*Every Pyramid* ABCD *having a triangular baſe, may be divided into two pyramides* AEGH, HIKC, *equal, and like one to the other, having baſes triangular, and like to the whole* ABDC; *and into two equal priſms,* BFGEIH, FGDIHK; *which two priſms are greater than the half of the whole pyramid* ABDC.

Divide the ſides of the pyramid into two parts at the
points E, F, G, H, I, K, and join the right-lines EF, FG,
GE, EI, IF, FK, KG, GH, HE. Becauſe the ſides of
the pyramid are proportionally cut (*a*) thence HI, AB, a 2.6.
and GF, AB; and IF, DC; and HG, DC, &c. are
parallels, and conſequently HI, FG; and GH, FI are
alſo parallels, therefore it is apparent that the triangles
ABD, AEG, EBF, FDG, HIK, (*b*) are equiangular, and
that the four laſt are (*c*) equal: In like manner the triangles b 29. 1.
ACB, AHE, EIB, HIC, FGK are equiangular; and c 26. 1.
the four laſt are equal one to the other. Alſo the triangles
BFI, FDK, IKC, EGH; and laſtly, the triangles
AHG, GDK, HKC, EFI are like and equal. Moreover
the triangles, HIK to ADB, EGH to BDC, and EFI
to ADC, and FGK to ABC, (*d*) are parallel. From d 15. 11.
whence it evidently follows, firſt, that the pyramids
AEGH, HIKC are equal, and (*e*) like to the whole ABDC, e 10. *def.*
and to one another. Next, that the ſolids BFGEIH, 11.
FGDIHK are priſms, and that of equal heights, as being
placed between the parallel planes ABD, HIK, but
the baſe BFGE is (*f*) double of the baſe FDG, wherefore
the ſaid priſms are equal; whereof the one BFGEIH is
greater than the pyramid BEFI, that is, than AEGH, f 2 *ax.* 1.
the whole than its part; and conſequently the two g 40. 11.
priſms are greater than the two pyramids and ſo exceed
the half of the whole pyramid ABDC. *Which was
to be dem.*

## PROP. IV. *Fig.* 16, 17.

*If there are two pyramids* ABCD, EFGH, *of the ſame altitude, having triangular baſes* ABC, EFG; *and either of them be divided into two pyramids* (AILM, MNOD; *and* EPRS, STVH) *equal one to the other and like to the whole; and into two equal priſms* (IBKLMN, KLCNMO; *and* PFQRST, QRGTSV;) *and if in like manner either of theſe pyrs. made by the former diviſion be divided, and this be done continually; then as the baſe of one pyramid is to the baſe of the other pyramid, ſo are all the priſms which are in one pyramid, to all the priſms which are in the other pyramid, being equal in multitude*

For (applying the conſtruction of the precedent
prop.)BC :KC (*a*) : :FG : QG; (*b*) therefore the triangle a 15. 5.
ABC is to the like triangle LKC as EFG is to (*c*) the b 22. 6.
like RQG; therefore by permutation ABC : EFG (*d*) : : c 2. 6.&c.
LKC d 16. 5.

e *ſch.* 34. 11. LKC : RQG (*e*) : : the priſm KLCNMO : QRGTSV
f 1. 5. (for theſe are of equal altitude) (*f*) : : IBKLMN : PF-
g 12. 5. QRST; (*g*) wherefore the triangle ABC : EFG : : the
priſm KLCMNO + IBKLMN : the priſm QRGTSV +
PFQRST. *Which was to be dem.*

But if the pyramids MNOD, AILM; and EPRS, STVH be further divided; in like manner the four new priſms made hereby ſhall be to four produced before, as the baſes MNO and AIL are to the baſes STV, and EPR, that is, as LKC to RQG, or as ABC to EFG. (*h*) wherefore all the priſms of the pyramid ABCD are to all the priſms of the pyramid EFGH as the baſe ABC is to the baſe EFG. *Which was to be dem.*

## PROP. V. Plate VI. *Fig.* 18, 19.

*Pyramids* ABCD, EFGH, *being of the ſame altitude, and having triangular baſes* ABC, EFG, *are one to another as their baſes* ABC, EFG.

Let the triangle ABC : EFG : : ABCD : X. I ſay
X is equal to the pyramid EFGH. For if it be poſſible,
let X be ⊐ EFGH. and let the exceſs be Y. Divide
the pyramid EFGH into priſms and pyramids, and
a 1. 10. the other pyramids in like manner, (*a*) till the pyrs.
left EPRS, STVH, be leſs than the ſolid Y. Therefore
ſince the pyramid EFGH = X + Y, it is manifeſt
that the remaining priſms PFQRST, QRGTSV are
greater than the ſolid X. Conceive the pyramid ABCD
b 4. 12. divided after the ſame manner; (*b*) then will be the
c *hyp.* priſm IBKLMN + KLCNMO : PFQRST + QRGTSV
d 14. 5. : : ABC : EFG (*c*) : : the pyr. ABCD : X; (*d*) there-
fore X ⊏ the priſm PFQRST + QRGTSV; which is
contrary to that which was affirmed before.

Again, conceive X ⊏ the pyr. EFGH, and make the
pyr. EFGH : Y : : X : the pyr. ABCD (*e*) : : EFG :
e *hyp. and cor.* ABC. Becauſe EFGH (*f*) ⊐ X, (*g*) thence Y ⊐ the
4. 5. pyr. ABCD; which is ſhewn before to be impoſſible.
f *ſuppoſ.* Therefore I conclude, that X is equal to the pyr. EFGH.
g 14. 5. *Which was to be demonſtrated.*

PROP.

## PROP. VI. Plate VI. *Fig.* 20, 21.

*Pyramids* ABCDEF, GHIKLM, *being of the ſame altitude, and having polygonous baſes* ABCDE, GHIKL, *are to one another as their baſes* ABCDE, GHIKL. *are.*

Draw the right-lines AC, AD, GI, GK, then is the baſe ABC : ACD (*a*) : : the pyr. ABCF : ACDF; (*b*) therefore by compoſition, ABCD : ACD : : the pyr. ABCDF : ACDF; (*a*) but alſo ACD : ADE : : the pyr. ACDF : ADEF; (*c*) therefore by equality ABCD : ADE : : ABCDF : ADEF, and (*b*) thence by compoſition ABCDE : ADE : : the pyr. ABCDEF : ADEF; moreover ADE : GKL (*d*) : : the pyr. ADEF : GKLM; and as before, and inverſely GKL : GHIKL : : the pyr. GKLM : GHIKLM; (*c*) therefore again by equality ABCDE : GHIKL : : the pyr. ABCDEF : GHIKLM. *Which was to be demonſtrated.*

a 5. 12.
b 18 5.
c 22. 5.
d 5. 12.

If the baſes have not ſides of equal multitude, the demonſtration will proceed thus. The baſe (*Fig.* 20, 22.) ABC : GHI (*e*) : : the pyr. ABCF : GHIK (*e*) and ACD : GHI : : the pyr. ACDF : GHIK, (*f*) therefore the baſe ABCD : GHI : : the pyr. ABCDF : GHIK. (*e*) Moreover the baſe ADE : GHI : : the pyr. ADEF : GHIK, (*f*) therefore the baſe ABCDE : GHI : : the pyr. ABCDEF : GHIK.

e 5. 12.
f 24. 5.

## PROP. VII. *Fig.* 23.

*Every priſm,* ABCDEF, *having a triangular baſe, may be divided into three pyrs.* ACDF, ACBF, CDFE, *equal one to the other, and having triangular baſes.*

Draw the diameters of the parallelograms AC, CF, FD. Then the triangle ACB is (*a*)=ACD; (*b*) therefore the pyramids of equal height ACBF, ACDF, are equal. In like manner the pyr. DFAC=the pyr. DFEC, but ACDF and DFAC are one and the ſame pyr. (*c*) therefore the three pyramids ACBF, ACDF, DFEC, into which the priſm is divided, are equal one to the other. *Which was to be demonſtrated.*

a 34. 1.
b 5. 12.
c 1. *ax.* 1.

Hence, every pyramid is the third part of the priſm that has the ſame baſe and height with it, or every priſm is treble of the pyramid that has the ſame baſe and height with it.

For reſolve the polygonous priſm ABCDEGHIKF,
(*Fig.* 24.) into triangular priſms; and the pyr. ABCD-
a 7. 12. EH into triangular pyramids; (*a*) then all the parts of
b 1. 5. the priſm ſhall be treble to all the parts of the pyramids,
(*b*) conſequently the whole priſm ABCDEGHIKF is
treble to the whole pyr. ABCDEH. *Which was to be
demonſtrated.*

### PROP. VIII. Plate VI. *Fig.* 25, 26.

*Like pyramids* ABCD, EFGH, *which have triangular baſes* ABC, EFG *are in triplicate ratio of their homologous ſides* AC, EG.

a 27. 11. (*a*) Compleat the parallelpps. ABICDMKL, EFNG-
b 9. *def.* 11. HQOP, which (*b*) are like, and (*c*) ſextuple of the pyra-
c 28. 11. mids ABCD, EFGH; (*d*) and therefore the pyrs. have
*and* 7. 12. the ſame proportion to one another as the parallelps.
d 15. 5. have, that is, (*e*) triplicate of their homologous ſides.
e 33. 11.

### *Coroll.*

Hence, alſo like polygonous pyramids are in triplicate ratio of their homologous ſides; as may be eaſily prov'd by reſolving them into triangular pyramids.

### PROP. IX. *Fig.* 25, 26.

*In equal pyramids* ABCD, EFGH, *having triangular baſes* ABC, EFG, *the baſes and altitudes are reciprocal; And pyramids having triangular baſes, whoſe altitudes and baſes are reciprocal, are equal.*

1. *Hyp.* The compleated parallelpps. ABICDMKL,
a 28. 11. EFNGHQOP are (*a*) ſextuple of the equal pyramids
*and* 7. 12. ABCD, EFGH (each to each) and ſo equal one to the
b 34. 11. other; therefore as the altitude (H) : the altitude (D)
c 15. 5. (*b*) : : ABIC : EFNG; (*c*) : : ABC : EFG. *Which*
d *hyp.* *was to be demonſtrated.*

2. *Hyp.* The altitude (H) : the altitude (D.) (*d*) : :
e 15. 5. ABC : EFG (*e*) : : ABIC : EFNG (*f*) therefore the
f 34. 11. parallelpps, ABICDMKL, EFNGHQOP are equal;
g 6. *ax.* 1. (*g*) conſequently alſo the pyramids ABCD, EFGH being
ſubſextuple of the ſame, are equal. *Which was to be
demonſtrated.*

*The ſame is applicable to polygonous pyramids, for they may alſo in like manner be reduced to triangular.*

### *Coroll.*

*Whatſoever is dem. of pyramids in* prop. 6, 8, 9 *does likewiſe agree to any ſort of priſms; ſeeing they are triple of the pyramids that have the ſame baſe and altitude with them.* Therefore

1. The proportion of priſms of equal altitude is the ſame with that of their baſes.
2. The proportion of like priſms is triplicate of that of their homolgous ſides.
3. Equal priſms have their baſes and altitudes reciprocal; and priſms which are ſo reciprocal, are equal.

### *Schol.*

From what has been hitherto dem. the dimenſion of any priſms and pyramids may be collected.

(*a*) The ſolidity of a priſm is produced from the al- a 1. *cor.* 12.
titude multiplied into the baſe; (*b*) and therefore like- & *ſch.* 4.
wiſe that of a pyr. from the third part of the altitude 11.
multiplied into the baſe. b 7. 12.

### PROP. X, Plate VI. *Fig.* 27.

*Every Cone is the third part of a cylinder having the ſame baſe with it* ABCD, *and the altitude equal.*

If you deny it, then firſt let ſuch cylinder be more
than triple to the cone, and let the exceſs be E. A priſm
deſcribed on a ſquare in the circle ABCD (*a*) (*Fig.* 13, a *ſch.* 7. 4.
14.) is ſubduple of a priſm deſcribed upon a ſquare about *and cor.* 9.
the circle, being equal to it and the cylinder in height. 12.
Therefore a priſm upon the ſquare, ABCD exceeds the
half of the cylinder, and likewiſe a priſm upon the baſe
AFB, of equal height to the cylinders, (*b*) is greater than b *ſch.* 27. 3.
the half of the ſegment of the cyl. AFB, continue an *and cor.* 9.
equal biſection of the arches, and ſubſtract the priſms till 12.
the remaining ſegments of the cyl. namely, at AF, FB,
&*c.* become leſs than the ſolid E. Therefore the cyl.
— ſegm. AF, FB, &*c.* (the priſm on the baſe AFBGCH-
DI) (*c*) is greater than the cyl. — E (*d*) (the triple of the c 5. *ax.* 1.
cone) therefore the pyr. (*e*) a third part of the ſaid priſm d *hyp.*
(being placed on the ſame baſe, and of the ſame height) e *cor.* 7. 12.
is greater than the cone of equal height on the baſe
ABCD a circle, *i. e.* the part greater than the whole
*Which is abſurd.*

But if the cone be affirmed to be greater than the third part of the cyl. then let the exceſs be E. Subtract the pyrs. from the cone, as you did in the firſt part the priſms from the cyl. till ſome ſegments of the cone remain, ſuppoſe AF, FB, BG, *&c.* leſs than the ſolid E, therefore the cone — E (*f*) $\frac{1}{3}$ of the cyl.) ⊐ the pyr. AFBGCHDI (the cone — ſeg. AF, FB, *&c.*) therefore the priſm triple to the pyr. (*viz.* of equal height, and on the ſame baſe) is greater than the cyl. on the baſe ABCD, the part than the whole. *Which is abſurd.* Wherefore it muſt be granted, that the cyl is equal to triple of the cone. *Which was to be dem.*

## PROP. XI. Plate VI. *Fig.* 28. 29.

*Cylinders and Cones* ABCDK, EFGHM, *being of the ſame altitude, are to one another as their baſes* ABCD, EFGH *are.*

Let the circle ABCD : the circle EFGH : : the cone ABCDK : N. I ſay N is equal to the cone EFGHM.

For if it be poſſible, let N be ⊐ the cone EFGHM,
and let the exceſs be O. The preparation and argumen-
tation of the prec. prop. being ſuppoſed ; then ſhall O
be greater than the ſegments of the cone EP, PF, FQ,
*&c.* and ſo the ſolid N ⊐ the pyr. EPFQGRHSM. In
a 30. 3. *and* the circle ABCD (*a*) make a like polyg. fig. ATBVCX-
1 *poſt.* DY. Becauſe the pyr. ABVYK : the pyr. EFQSM (*b*) : :
b 6. 12. the polygon ATBVY : the polygon EPFQS (*c*) : : the
c *cor.* 2. 12. circle ABCD : the circle EFGH (*d*) : : the cone ABCDK
d *hyp.* : N (*e*) thence the pyr. EPFQGRHSM ſhall be ⊐ N,
e 14. 5. contrary to what was affirmed before. Again, conceive N
f *hyp. and by* ⊏ the cone EFGHM, and make the cone EFGHM :
*inverſion.* O : : N : the cone ABDCK (*f*) : : the circle EFGH :
g 14. 5. ABCD ; (*g*) therefore O ⊐ the cone ABCDK ; *which
is abſurd,* as appears by what is ſhewn in the firſt part.

Therefore rather admit ABCD : EFGH : : the cone ABCDK : EFGHM. *Which was to be dem.*

The ſame may be dem. of cylinders, if cylinders and priſms be conceived in the place of cones and pyramids, therefore, *&c.*

### *Schol.*

*Hence, is gathered the dimenſion of all ſorts of cylinders
and cones.* The ſolidity of a right cyl. is produced from
a 1. *prop de* the circular baſe (*a*) (the dimenſion whereof is to be
*demenſion. cir.* learnt out of *Archimedes*) multiplied into the height ; (*b*)
b 11. 12. whence in like manner that of every cylinder.

(*c*) Therefore

(*e*) Therefore the ſolidity of a cone is produced from the third part of the altitude multiplied into the baſe.

e 10. 12.

## PROP. XII. Plate VI. *Fig.* 30, 31.

*Like comes and cylinders* ABCDK, EFGHM, *are in triplicate ratio of that of the diameters* TX, PR, *of their baſes* ABCD, EFGH.

Let the cone A have to N a triplicate ratio of TX to PR. I ſay N is = the cone EFGHM. For if it be poſſible let N be ⊐ EFGHM, and let the exceſs be O, therefore N ⊐ the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right-lines VK, CK, VI, CI, and QM, GM, QL, GL. Becauſe the cones are like, (*a*) thence VI : IK : : QL : LM; but the angles VIK, QLM (*b*) are right-angles, (*c*) therefore the triangles VIK, QLM are equiangular, (*d*) whence VC : VI : : QG : QL. alſo VI : VK : : QL : QM. therefore by equality VC : VK : : QG : QM. (*e*) moreover VK : CK : : QM : MG; therefore again by equality VC : CK : : QG : GM; (*f*) therefore the triangles VKC, QMG are like; and by a like way of reaſoning the other triangles of this pyr. are like to the other of that, (*g*) wherefore the pyrs. themſelves are like. (*h*) But theſe are in triplicate proportion of that of VC to QG, (*k*) that is, of VI to QL, (*l*) or TX to PR; (*m*) therefore the pyr. ATBVCXDYK : the pyr. EPFQGRHSM : : the cone ABCDK : N; (*n*) whence the pyr. EPFQGRHSM ⊐ N. which is repugnant to what was affirmed before.

a 24. *def.* 11
b 18. *def.* 11.
c 6. 6.
d 4. 6.
e 7. 5.
f 5. 6.
g 9. *def.* 11.
h *cor.* 8. 12.
k 4. 6.
l 15. 5.
m *hyp.* *and* 11. 5.
n 14. 5.

Again, take N ⊏ the cone EFGHM, make the cone EFGHM : O : : N : the cone ABCDK (*o*) : : the pyr, EPRM : ATCK (*p*) : : GQ : VC thrice : : (*q*) PR : TX thrice, therefore O (*r*) is ⊐ ABCDK. which was before ſhewn to be repugnant. Wherefoe N = the cone EFGHM. *Which was to be demonſtrated.*

o *before* & *inverſly.*
p *cor.* 8. 12
q 4. 6.
r 14. 5.

But foraſmuch as what proportion ſoever cones have, alſo cylinders, being triple of them, have the ſame; therefore cyl. ſhall be to cyl. in triplicate ratio of the diameters of their baſes.

## PROP. XIII. Plate VI. *Fig* 32.

*If a cyl.* ABCD *be divided by a plane* EF *parallel to the oppoſite planes* BC, AD, *then as one of cyl.* AEFD *is to the other cyl.* EBCF, *ſo is the axis* GI *to the axis* IH.

a 3. 1. The axis being produced, (*a*) take GK = GI, and
HL = IH = LM, and conceive planes drawn at the
b 11. 12. points K, L, M, parallel to the circles AD, BC, (*b*) therefore the cyl. ED = the cyl. AN, and the cyl. EC (*b*) = BO (*b*) = OP; therefore the cyl. EN is the ſame multiple of the cyl. ED as the axis IK is of the axis IG, and in like manner the cyl. EP is the ſame multiple of the cyl. BF, as the axis IM is of the axis IH; but as IK is
c 11. 12. =, ⊏, ⊐ IM, (*c*) ſo is the cyl. EN =, ⊏, ⊐ EP
d 6. *def.* 5. (*d*) therefore the cyl. AEFD : the cyl. EBCF. : GI :IH.
*Which was to be dem.*

## PROP. XIV. *Fig.* 33.

*Cones* AEB, CFD, *and cylinders* AH, CK, *ſtanding upon equal baſes* AB, CD, *are to one another, as their altitudes* ME, NF

The cyl. HA, and the axis EM, being produced, take ML = FN; and thro' the point L draw a plane
a 11. 12. parallel to the baſe AB, (*a*) then ſhall the cyl. AP
b 13. 12. be = CK, (*b*) but the cyl. AH : AP (CK) : : ME : ML
(NF.) *Which was to be dem.*

* *apply* 9. & 7. 12. The ſame may be affirmed of cones which are ſubtriple of cylinders; * as alſo of priſms and pyramids.

## PROP. XV. *Fig.* 34.

*In equal cones* BAC, EDF *and cylinders* BH, EK, *the baſes and altitudes are reciproc.* (BC : EF : : MD : LA) *And cones and cylinders, whoſe baſes and altitudes are reciprocal, are equal one to the other.*

If the altitudes be equal then the baſes are equal too, and the thing is evident. If unequal, then take away MO = LA.

a 14. 12. 1. *Hyp.* Then is MD : MO(*a*)(LA)(*b*): : the cyl. EK
b *conſtr.* : (*c*) (BH) EQ(*d*) : : the cir. BC : EF. *Which was to*
c *hyp.* *be dem.*
d 11. 12.

2. *Hyp.* BC : EF (*e*) : : DM : OM (LA) (*f*) : : the e *hyp.*
cyl. EK : EQ (*g*) : : BC : EF (*h*) : : BH : EQ (*k*) There- f 14. 12.
fore the cyl. EK : = BH. *Which was to be dem.* g 11. 5.
The ſame argument may be uſed for cones. h 11. 12.
k 9. 5.

## PROP. XVI. Plate VI. *Fig.* 35.

*Two unequal circles* ABCG, DEF, *having the ſame center* M, *to inſcribe in the greater circle* ABCG *a polygonous figure of equal and even ſides, which ſhall not touch the leſſer circle* DEF.

Thro' the center M draw the line AC cutting the cir.
DEF in F, from whence raiſe a perpendiculer FH, (*a*) a 30. 3.
divide the ſemicircle ABC into two equal parts; and
the half thereof BC alſo; and ſo do continually (*b*) till b 1. 10.
the arch IC becomes leſs than the arch HC; from I let
fall the perpendicular IL. It is manifeſt that the arch
IC meaſures the whole circle, and that the number of
arches is even, and ſo that the ſubtended-line IC is the
ſide (*c*) of the polygon that may be inſcribed without c *ſch.* 16. 4.
touching the leſſer circle DEF. For HG (*d*) touches d *cor.* 16. 3.
the circle DEF, (*e*) to which IK is parallel, and placed e 28. 1.
outwardly; (*f*) wherefore IK does not touch the cir- f 34. *def.* 1.
cle DEF; much leſs do CI, CK, and the other ſides
of the polygon more remote from the center. *Which was to be done.*

*Coroll.*

Obſerve that IK touches not the circle DEF.

## PROP. XVII. *Fig.* 36.

*Two ſpheres* ABCV, EFGH, *conſiſting about the ſame center* D, *being given, to inſcribe a ſolid of many ſides* (*or Polyedron*) *in the greater ſphere* ABCV, *which ſhall not touch the ſuperficies of the leſſer ſphere* EFGH.

Let both the ſpheres be cut by a plane paſſing thro'
the center, making the circles EFGH, ABCV; and the
diameters AC, BV drawn, cutting perpendicularly. In
the circle ABCV, (*a*) inſcribe the equilateral poly- a 16. 12.
gon VMLNC, &c. not touching the circle EFGH: Then
draw the diameters N $\alpha$, and erect DO perpendicular
to the plane ABC thro' DO, and thro' the diameters
AC, N $\alpha$ conceive planes DOC, DON erected, which
ſhall be (*b*) perpendicular to the circle ABCV, and ſo in b 18. 11.
the

c *cor.* 33. 6. the ſuperficies of the ſphere make (*c*) the quadrants DOC,
DON. In which let the right-lines CP, PQ, QR, RO,
d 4. 1. NS, ST, T$\gamma$., $\gamma$O (*d*) be fitted, equal, and of equal multitude with CN, NL, &c. make the ſame conſtruction in
the other quadrants OL, OM, &c. and in the whole
ſphere. Then I ſay the thing required is done.

From the points P, S, to the plane ABCV draw the
e 38. 11. perpendiculars PX, SY, (*e*) which ſhall fall on the ſections
f 12. *ax.* AC, N$\alpha$. Therefore becauſe both (*f* the right-angles
g 27. 3. PXC, SYN, (*g*) and PCX, SNY inſiſting on (*h*) equal
h 32. 1. circumferences, (*f*) are equal, the triangles alſo PCX,
k *conſtr.* SNY (*h*) are equiangular. Wherefore ſince PC (*k*) $=$ SN,
l 26. 1. (*l*) alſo is PX $=$ SY, (*l*) and XC $=$ YN; (*m*) whence DX
m 3. *ax.* 1. $=$ DY, (*n*) and therefore DX : XC : : DY : YN; (*o*)
n 7. 5. therefore YX, NC are parallels, but becauſe PX, SY are
o 2. 6. equal, and ſince being perpendicular to the ſame plane
p 6. 11. ABCV, they are alſo (*p*) parallels, (*q*) therefore YX, SP.
q 33. 1. ſhall be equal and parallel, (*r*) whence SP, NC, are pa-
r 9. 11. rallel one to the other; and ſo the (*ſ*) quadrilateral NC-
ſ 7. 11. PS, and for the ſame reaſon SPQT, TQRG, as alſo the
t 2. 11. (*t*) triangle $\gamma$RO are ſo many planes. In like manner
the whole ſphere may be ſhewn full of ſuch quadrilaterals
and triangles, wherefore the figure inſcribed is a polyedron.

u 11. 11. From the center D (*u*) draw DZ perpendicular to the
plane NCPS; and join ZN, ZC, ZS, ZP. Becauſe DN
x 4. 6. : NC (*x*) : : DY : YX, thence NC is (*y*) $\sqsubset$ YX (SP.) and
y 14. 5. in like manner SP $\sqsubset$ TQ, and TQ $\sqsubset$ $\gamma$R. And becauſe
z 3 *def.* 11. the angles DZC, DZN, DZS, DZP (*z*) are right, and
a 15. *def.* 1. the ſides DC, DN, DS, DP, (*a*) equal, and DZ common,
b 47. 1. (*b*) thence ZC, ZN, ZS, ZP are equal one to the other;
c 15. *def.* 1. and conſequently about the quadrilateral NCPS, (*c*) a circle may be deſcribed, in which (becauſe NS, NC, CP,
d *conſtr.* are (*d*) equal, and NC $\sqsubset$ SP) NC (*e*) ſubtends more than
e 28. 3. a quadrant, (*f*) therefore the angle NZC at the center is
f 33. 6. obtuſe, (*g*) therefore NCq $\sqsubset$ 2 ZCq (ZCq + ZNq).
g 12. 2. Let NI be drawn perpendicular to AC, therefore ſince
h 32. 1. the angle ADN (*h*) DNC + DCN (*k*) is obtuſe, the half
k 9. *ax.* 1. of it DCN ſhall be greater than the half of a right an-
l 5. 1. gle; and ſo that which remains of the right-angle CNI
n 19. 1. ſhall be leſs than it, (*n*) whence IN $\sqsubset$ IC, therefore
o 47. 1. NCq (NIq + ICq) (*o*) $\sqsupset$ 2 INq. therefore IN $\sqsubset$ ZC,
p 47. 1. and conſequently DZ (*p*) $\sqsubset$ DI; but the point I is (*q*) with-
q *cor.* 16. out the ſphere EFGH, and ſo, much more, the point Z.
12. wherefore the plane NCPS, (of which (*r*) the neareſt point
r 47. 1. to the center is Z.) does not touch the ſphere, EFGH.
And

And if a perpendicular Dſ be drawn to the plane SPQT, the point ſ, and ſo alſo the plane SPQT is yet further removed from the center, which is alſo true of the other planes of the polyedron. Thererefore the polyedron ORQPCN, *&c.* inſcribed in the greater ſphere, does not touch the leſſer. *Which was to be done.*

*Coroll.*

Hence it follows, *that if in any other ſphere a ſolid polyedron, like to the abovesaid ſolid polyedron, be inſcribed, the proportion of the polyedron in one ſphere to the polyedron in the other is triplicate of that of the diameters of the Spheres.*

For if right-lines be drawn from the centers of the ſpheres to all the angles of the baſes of the ſaid polyedrons, then the polyedrons will be divided into pyrs. equal in number and like; whoſe homo. ſides are ſemidiameters of the ſpheres; as appears, if the leſſer of theſe ſpheres be conceived deſcribed within the greater about the ſame center. For the right-lines drawn from the center of the ſphere to the angles of the baſes will agree one to the other by reaſon of the likeneſs of the baſes; and ſo will like pyramids be made. Wherefore ſince every pyr. in one ſphere to every pyr. like it in the other ſphere (*a*) has proportion triplicate to that of the homologous ſides, that is, of the ſemidiameter of the ſpheres; and (*b*) as one pyr. is to one pyr. ſo all the pyrs. that is, the ſolid polyedron compoſed of theſe, are to all the pyrs. that is, the ſolid polyedron compoſed of the others; therefore the polyedron of one ſphere ſhall have to the polyedron of the other ſphere, proportion triplicate to that of the ſemidiameters, (*c*) and ſo of the diameters of the ſpheres.

a *cor.*8. 12.
b 12. 5.
c 15. 5.

## PROP. XVIII. Plate VI. *Fig.* 38.

*Spheres* BAC, EDF, *are in triplicate ratio of their diameters* BC, EF.

Let the ſphere BAC be to the ſphere G in tripli. proportion of that of the diameter BC to the diameter EF. I ſay G = EDF. For if it be poſſible, let G be ⊐ EDF. and conceive the ſphere G concentrical with EDF. In the ſphere EDF (*a*) inſcribe a polyedron not touching the ſphere G, and a like polyedron in the ſphere BAC. Theſe poleydrons (*b*) are in triplicate proportion of the diameters

a 17. 12.
b *cor.*17. 12.

c *hyp.* diameters BC, EF, (*c*) that is, of the ſphere BAC to G.
d 14. 5. (*d*) conſequently the ſphere G is greater than the polyedron inſcribed in the ſphere EDF, the part than the whole.

Again, if it be poſſible, let the ſphere G be $\sqsubset$ EDF. and as the ſphere EDF is to another ſphere H, ſo let G
e *hyp. inverſe.* be to BAC, (*e*) that is, in triplicate proportion of the dia-
f 14. 5. meter EF to BC, therefore ſince BAC (*f*) $\sqsubset$ H, we ſhall incur the abſurdity of the firſt part, wherefore rather the ſphere G $=$ EDF. *Which was to be demonſtrated.*

*Coroll.*

Hence, as one ſphere is to another ſphere, ſo is a polyedron deſcribed in that to a like polyedron deſcribed in this.

*The End of the Twelfth* BOOK.

# The THIRTEENTH BOOK

# OF

# EUCLIDE's

# ELEMENTS.

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## PROPOSITION I. Plate VI. *Fig.* 39.

*IF a right-line* z *be divided according to extreme and mean proportion* (z : a : : a : e.) *the ſquare of the half of the whole line* z, *and of the greater ſegment* a, *as one line, is quintuple to that which is deſcribed of half of that whole line* z.

I ſay Q. a $+\frac{1}{2}$ z = 5 Q: $\frac{1}{2}$ z. (*a*) that is aa $+\frac{1}{4}$ zz $+$ za = zz $+\frac{1}{4}$ zz (*b*) or aa $+$ za = zz. For ze $+$ za (*c*) = zz. and ze (*d*) = aa. (*e*) therefore aa $+$ za = zz. *Which was to be demonſtrated.*

a 4. 2.
b 3. *ax.* 1.
c 2. 2.
d *hyp.* *and* 16. 6.
e 2. *ax.* & 1. *ax.*

## PROP. II. *Fig.* 39.

*If a right-line* $\frac{1}{2}$ z $+$ a *be in power quintuple to a ſegment of it ſelf* $\frac{1}{2}$ z. *the line double of the ſaid ſegment* (z) *being divided according to extreme and mean proportion, the greater ſegment is* (a) *the other part of the right-line at firſt given* $\frac{1}{2}$ z $+$ a.

I ſay z : a : : a : e. Becauſe by the hyp. * aa $+\frac{1}{4}$ zz $+$ za = zz $+\frac{1}{4}$ zz; or aa $+$ za = zz (*a*) = ze $+$ za, (*b*) thence ſhall aa = ze. (*c*) wherefore z: a : : a : e. *Which was to be demonſtrated.*

* 4. 2.
a 2. 2.
b 3. *ax.*
c [illegible]

## PROP. III. Plate VI. *Fig* 40.

*If a right-line* z *be divided according to extreme and mean proportion* (z : a : : a : e.) *the line made of the leſs ſegment* e *and half of the greater ſegment* a, *is in power quintuple to the ſquare, which is deſcribed of the half line of the greateſt ſegment* a.

a 4. 2. I ſay $Q : e + \frac{1}{2} a = 5\, Q : \frac{1}{2} a$ : (*a*) that is $ee + \frac{1}{4}$
b 3. *ax.* $aa + ea = aa + \frac{1}{4} aa$. (*b*) or $ee + ea = aa$. For ee
c 3. 2. $+ ea$ (*c*) $= ze$ (*d*) $= aa$. *Which was to be demonſtrated.*
d *hyp. and* 17. 6.

## PROP. IV. *Fig.* 41.

*If a right-line* z *be cut according to extreme and mean proportion* (z : a : : a : e.) *the ſquare made of the whole line* z, *and that made of the leſſer ſegment* e, *both together, are triple of the ſquare made of the greater ſegment* a.

a 4. 1. I ſay $zz + ee = 3\, aa$. (*a*) or $aa + ee + 2\, ae + ee$
b 3. 2. $= 3\, aa$. For $ae + ee$ (*b*) $= ze$ (*c*) $= aa$. (*d*) therefore $aa$
c 17. 6. $+ 2\, ae + 2\, ee = 3\, aa$. *Which was to be demonſtrated.*
d 2. *ax.*

## PROP. V.

D——A——C——B *If a right-line* AB *be cut according to extreme and mean proportion in* C, *and a line* AD, *equal to the greater ſegment* BC, *added to it, the whole right-line* DB *is divided according to extreme and mean proportion; and the greater ſegment is the right-line* AB *given at the beginning.*

a *hyp.* For becauſe AB : AD. (*a*) : : AC : CB. and by inverſion AD : AB : : CB : AC. therefore by compoſition DB : AB : : AB : AC (AD.) *Which was to be demonſtrated.*

### *Schol.*

But if BD : BA : : BA : AD. then ſhall be BA : AD. : : AD : BA — AD. For by dividing BD—BA (AD) : BA : : BA — AD : AD. therefore inverſely BA : AD : : AD : BA — AD.

## PROP. VI.

D——A——C——B *If a rational right-line* AB *be cut according to extreame and mean proportion in* C, *either of the ſegments* (AC, CB) *is an irrational-line of that kind which is called apotome or reſidual.*

To the greater ſegment AC (*a*) add AD = $\frac{1}{2}$ AB. (*b*) a 3. 1.
therefore DCq = 5 DAq. (*c*) therefore DCq $\sqcap\!\!\sqcup$ DAq. b 1. 13.
conſequently (*d*) ſince AB, (*e*) and ſo the half thereof c 6. 10.
DA are $\acute{\rho}$, likewiſe DC is $\acute{\rho}$. But becauſe 5 : 1 : : not d *hyp.*
Q : Q. (*f*) thence is DC $\sqcup\!\!\sqcup$ DA. (*g*) therefore DC—AD, e *ſch.* 12. 10.
that is, AC, is a reſidual-line. Further, becauſe ACq f 9. 10.
(*h*) = AB×BC, and AB is $\acute{\rho}$, (*i*) likewiſe BC is a reſidual-line. g 74. 10.
*Which was to be demonſtrated.* h 17. 6.
i 98. 10.

## PROP. VII. Plate VI. *Fig.* 42.

*If three angles of an equilateral pentagon* ABCDE, *whether they follow in order,* (EAB, ABC, BCD,) *or not,* (EAB, BCD, CDE) *are equal, the pentagon* ABCDE *ſhall be equiangular.*

Let the right-lines BE, AC, BD, be ſubtended to the equal angles in order.

Becauſe the ſides EA, AB, BC, CD, and the included
angles (*a*) are equal, (*b*) therefore ſhall the baſes BE, AC, a *hyp.*
BD, (*c*) and the angles AEB, ABE, BAC, BCA, be equal. b 4. 1.
(*d*) Wherefore BF = FA, (*e*) and conſequently FC = c 4. *and* 5.
FE; therefore the triangles FCD, FED, are equilateral 1.
one to the other: (*f*) whence the angle FCD = FED. (*g*) d 6. 1.
conſequently the angle AED = BCD. In like manner e 3. *ax.* 1.
the angle CDE is equal to the reſt; wherefore the pen- f 8. 1.
tagon is equiangular. *Which was to be demonſtrated.* g 2. *ax.* 1.

But if the angles EAB, BCD, CDE, which are not
in order, be ſuppoſed equal, (*h*) then ſhall the angle AEB h 4. 1.
= BDC, and BE = BD. (*k*) and thence the angle BED k 5. 1.
= BDE. (*l*) conſequently the whole angle AED = CDE, l 2. *ax.*
therefore becauſe the angles A, E, D, in order, are equal, as before, the pentagon ſhall be equiangular. *Which was to be demonſtrated.*

## P R O P. VIII. Plate VI. *Fig.* 43.

*If in an equilateral and equiangular pentagon* ABCDE, *two right-lines* BD, CE, *ſubtend two angles* BCD, CDE *following in order, thoſe lines cut one another according to extreme and mean proportion; and their greater ſegments* BF *or* EF *are equal to the ſide of the pentagon* BC.

a 14. 4. (*a*) Deſcribe about the pentagon the circle ABD. (*b*)
b 28. 3. The arch ED is = BC, (*c*) therefore the angle FCD =
c 27. 3. FDC. (*d*) therefore the angle BFC = 2 FCD (FCD +
d 32. 1. FDC.) But the arch BAE is = 2 ED, and conſequently
e 33. 6. the angle BCF (*e*) = 2 FCD = BFC. (*f*) wherefore
f 6. 1. BF = BC. *Which was to be demonſtrated.* Moreover,
g 27. 3. becauſe the triangles BCD, FCD, are (*g*) equiangular. (*h*)
h 4. 6. therefore BD : DC (BF) : : CD : (BF) FD. and likewiſe
EC : EF : : EF : FC. *Which was to be demonſtrated.*

## P R O P. IX. *Fig.* 44.

*If the ſide of an Hexagon* BE, *and the ſide of a Decagon* AB *both deſcribed in the ſame circle* ABC, *be added together, the whole right-line* AE *is cut according to extreme and mean proportion* (AE : BE : : BE: AB.) *and the greater ſegment therefore is the ſide of the Hexagon* BE.

Draw the diameter ADC, and join the right-lines
a *hyp. and* DB, DE. Becauſe the angle BDC (*a*) = 4 BDA and the
27. 3. angle BDC (*b*) = 2 DBA (DAB + DBA) thence ſhall
b 32. 1. DBA (*b*) (DBE + BED) (*c*) be = 2 BDA (*d*) = 2 BDE,
c 7. *ax.* 1. whence the angle DBA or DAB (*e*) = ADE. There-
d 5. 1. fore the triangles ADE, ADB, are equiangular : (*f*)
e 1. *ax.* 1. wherefore AE : AD (*g*) (BE) : : AD : (BE) AB.
f 4. 6. *Which was to be demonſtrated.*
g *cor.* 15. 4.

### *Coroll.*

Hence, if the ſide of a hexagon in a circle be cut according to extreme and mean proportion; the greater ſegment thereof ſhall be the ſide of the decagon in the ſame circle.

## PROP. X. Plate VII. *Fig.* 1,

*If an equilateral Pentagon* ABCDE *be inſcribed in a circle* ABCE, *the ſide of the pentagon* AB *containeth in power both the ſide of a hexagon* FB, *and the ſide of a decagon* AH *inſcribed in the ſame circle.*

Draw the diameter AG, and biſect the arch AH in K, and draw FK, FH, FB, BH, HM.

The ſemicircle AG — the arch AC (*a*) $=$ AG — AD. that is, the arch CG $=$ GD (*b*) $=$ AH $=$ HB. therefore the arch BCG $= 2$ BHK; (*c*) and ſo the angle BFG $= 2$ BFK. (*d*) but the angle BFG $= 2$ BAG; (*e*) therefore the angle BFK $=$ BAG. Wherefore the triangles BFM, FAB, (*f*) are equilangular; (*g*) whence AB : BF : : BF : BM; (*h*) therefore $AB \times BM = BFq$. Moreover the angle AFK (*k*) $=$ HFK, and FA$=$FH; (*m*) wherefore AL$=$LH, (*m*) and the angles FLA, FLH are equal, and ſo right-angles, therefore the angle LHM (*m*) $=$ LAM (*n*) $=$ HBA; therefore the triangles AHB, AMH, (*o*) are equiangular; wherefore AB : AH : : AH : AM; (*q*) therefore $AB \times AM = AHq$. Since therefore ABq (*r*) $= AB \times BM + AB \times AM$, (*ſ*) thence $ABq = BFq + AHq$. *Which was to be demonſtrated.*

a 28. 3. & 3 *ax.*
b *hyp.* *and* 7. *ax.*
c 33. 6.
d 20. 3.
e 1. *ax.* 1.
f 32. 1.
g 4. 6.
h 17. 6.
k 27. 3.
m 4. 1.
n 27. 3.
o 32. 1.
p 4. 6.
q 17. 6.
r 2. 2.
ſ 2. *ax.*

### *Coroll.*

1. Hence, a right-line (FK) which being drawn from the center (F) divides an arch (HA) into two equal ſegments, does alſo divide the right-line (HA) ſubtending that arch perpendicularly into two equal ſegments.

2. The diameter of a circle (AG.) drawn from any angle (A) of a pentagon, does divide equally in two, both the arch (CD,) which the ſide of the pentagon oppoſite to that angle ſubtends, and alſo the oppoſite ſide it ſelf (CD) and that perpendicularly.

### *Schol. Fig.* 2.

*Here, according to our promiſe, we ſhall lay down a ready praxis of the* 11th prop. *of the* 4th *Book.*

### *Problem*

*To find the ſide of a pentagon to be inſcribed in a circle* ADB.

Draw the diameter AB, to which erect a perpendicular C D at the center C, divide CB equally in E, and make EF = ED, then DF ſhall be the ſide of the pentagon.

a 6. 2.
b conſtr.
c 47. 1.
d 3. ax.
e 17. 6.
f 9. 13.
g 10. 13.
h 47. 1.

For BF × FC + ECq (*a*) = EFq (*b*) EDq (*c*) = DCq + ECq. (*d*) therefore BF × FC = DCq or ECq. (*e*) wherefore BF : EC : : BC : FC; therefore ſince BC is the ſide of a hexagon, (*f*) FC ſhall be the ſide of a decagon. Conſequently DF (*h*) = $\sqrt{DCq+FCq}$ (*g*) is the ſide of a pentagon. *Which was to be done.*

## PROP. XI. Plate VII. *Fig.* 3.

*If in a circle* ABCD, *whoſe diameter* AG, *is rational, an equilateral pentagon be inſcribed* ABCDE; *the ſide of the pentagon* AB *is an irrational-line of that kind which is called a minor-line.*

* 10. 6.

Draw the diameter BFH, and the right-lines AC, AH; and * make FL = ¼ of the radius FH; and CM = ¼ CA.

a cor. 10. 13.
b 32. 1.
c 4. 6.
d 15. 5.
e 18. 5.
f 22. 6.
g 1. 13.
h 9. 10.
k 74. 10.
l 9. 10.
* 4 def. 85. 10.
m cor. 8. 6. and 17. 6.
n 95. 10.

Becauſe the angles AKF, AIC, are (*a*) right-angles, and CAI common, the triangles AKF, AIC, are (*b*) equiangular: (*c*) therefore CI : FK (*c*) : : CA : FA (FB; *d*) : : CM : FL; therefore by permutation FK : FL : : CI : CM (*d*) : : CD : CK (2 CM) and ſo by (*e*) compoſition CD + CK : CK : : KL : FL; (*f*) conſequently Q: CD + CK (*g*) (5 CKq) : CKq : : KLq : FLq. therefore KLq = 5 FLq. wherefore if BH (ρ̊) be taken 8, FH ſhall be 4. FL 1, and FLq 1, BL 5, and BLq 25, KLq 5. by which it appears that BL and KL are ρ̊ (*h*) ⊐, (*k*) and ſo BK is a reſidual and KL its congruent or adjoining-line; but ſince BLq − KLq. = 20. (*l*) thence BL ⊏ $\sqrt{BLq-KLq}$ * whence BK ſhall be a fourth reſidual-line. Therefore becauſe ABq (*m*) is = HB × BK, (*n*) ſhall AB be a minor-line. *Which was to be demonſtrated.*

## PROP. XII. *Fig.* 4.

*If in a circle* ABEC *an equilateral triangle* ABC *be inſcribed, the ſide of that triangle* AB *is in power triple to the line* AD *drawn from* D *the center of the circle to the circumference,*

The

Plate VI. *Facing Pag. 234.*

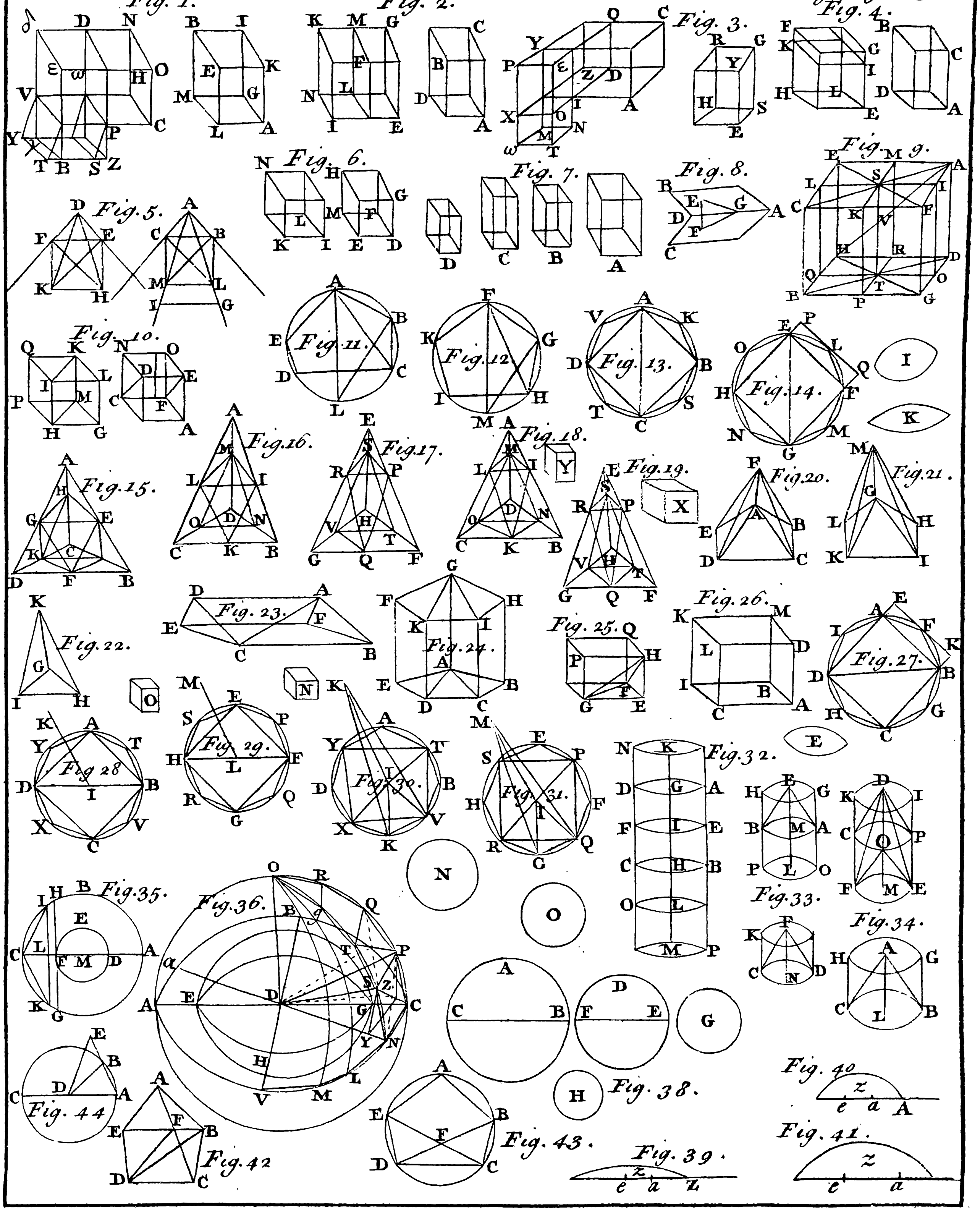
Fig. 1. δ D N ε ω H O V C Y P T B S Z
B I E K M G L A
Fig. 2. K M G F L N I E
C B D A
Q C Y P ε Z D I X O A N ω M T
Fig. 3. R G Y H S E
Fig. 4. F B K G I H L D E C A
Fig. 5. D F E K H
A C B M L I G
N Fig. 6. H G L M F K I E D
Fig. 7. D C B A
Fig. 8. B E G D F A C
Fig. 9. E M A L S I C K V F H R D O B P T G
Fig. 10. Q K N O P I L D E M C F H G A
Fig. 11. A B E D C L
Fig. 12. F K G I H M
Fig. 13. A V K D B T S C
Fig. 14. E P L O Q H F N M G
I
K
Fig. 15. A H G E K C D F B
Fig. 16. A M L I O D N C K B
Fig. 17. E S R P V H T G Q F
Fig. 18. A M L I Y O D N C K B
Fig. 19. E S R P X V H T G Q F
Fig. 20. F A E B D C
Fig. 21. M G L H K I
Fig. 22. K G I H
O
Fig. 23. D A E F C B
N
Fig. 24. G F H K I A E B D C
Fig. 25. Q P H F G E
Fig. 26. K M L D I B A C
Fig. 27. A E I F K D B H G C
E
Fig. 28. K A Y T D I B X V C
Fig. 29. M E S P H F L R Q G
Fig. 30. K A Y T D I B X V K
Fig. 31. M E S P H F I R Q G
N
O
Fig. 32. N K D G A F I E C H B O L M P
Fig. 33. H E G B M A P L O K F C N D
K D I C O P F M E
Fig. 34. H A G C L B
Fig. 35. I H B E C L F M D A K G
Fig. 36. O R Q B q T P a A E D S Z C G Y N H V M L
A C B
D F E
G
H
Fig. 38.
Fig. 39. z e a Z
Fig. 40. z e a A
Fig. 41. z e a
Fig. 42. A F E B D C
Fig. 43. A E B F D C
Fig. 44. E B C D A

The diameter being extended to E, draw BE. Be-
cauſe the arch BE (*a*) $= EC$, the arch BE is the ſixth a *cor.* 10.
part of the circumference, (*b*) therefore $BE = DE$. 13.
hence $AEq$ (*c*) $= 4\ DEq\ (4\ BEq)$ (*d*) $= ABq + BEq$ b *cor.* 15. 4.
$+ ADq.$) (*e*) conſequently $ABq = 3\ ADq$. *Which was* c 4. 2.
*to be dem.* d 47. 1.
e 3. *ax.* 1.

*Coroll.*

1. $AEq : ABq :: 4 : 3$.
2. $ABq : AFq :: 4 : 3$. (*f*) For $ABq : AFq :: AEq :$ f *cor.* 8. 6.
$ABq$. *and* 22. 6.
3. $DF = FE$. For the triangle EBD (*g*) is equi- g *cor* 15. 4.
lateral, (*h*) and BF perpendicular to ED: (*h*) there- h *cor.* 3. 3.
fore $EF = FD$.
4. Hence, $AF = DE + DF = 3\ DF$.

## PROP. XIII. Plate VII. *Fig.* 5, 6.

*To deſcribe a pyramid* EGFI, *and comprehend it in a ſphere given: and to demonſtrate that the diameters of the ſphere* AB *is in power ſeſquialter of the ſide* EF *of the pyramid* EGFI.

About AB deſcribe the ſemicircle ADB; (*a*) and let a 10. 6.
$AC$ be $= 2\ CB$. From the point C erect the perpendi-
cular CD, and join AD, DB, then at the interval of the
radius $HE = CD$ deſcribe the circle HEFG. (*b*) wherein b *cor.* 15. 4.
inſcribe the equilateral triangle EFG; from H (*c*) erect IH c 12. 11.
$= CA$ perpendicular to the plane EFG, produce IH to
K, (*d*) ſo that $IK = AB$; and join the right-lines IE, IF, d 3. 1.
IG. Then EFGI ſhall be the pyramid requried.

For becauſe the angles ACD, IHE, IHF, IHG, (*e*) are e *conſtr.*
right-angles; and CD, HE, HF, HG (*e*) equal, (*e*) and
$IH = AC$; (*f*) therefore AD, IE, IF, IG, ſhall be equal f 4. 1.
among themſelves. But becanſe $AC\ (2\ CB,) : CB$ (*g*) $::$ g 20. 6.
$ACq : CDq$. thence ſhall $ACq = 2\ CDq$; therefore $ADq$ h 2. *ax.*
(*f*) $= ACq + CDq$ (*h*) $= 3\ CDq = 3\ HEq$ (*k*) $= FFq$; k 12. 13.
(*l*) therefore AD, EF, IE, IF, IG are equal, and ſo the l 1. *ax.* 1.
pyramid EFGI is equilateral. But if the point C be placed
upon H, and AC upon HI, the right-lines AB, IK, (*m*) m 8. *ax.* 1.
ſhall agree, as being equal. Wherefore the ſemicircle
ABD being drawn about the axis AB or IK (*n*) n 15. *def.* 1
ſhall paſs by the points E, F, G, * and ſo the pyramid EF- * 31. *def.*
GI ſhall be inſcribed in a ſphere. *Which was to be done.* 11.

Alſo

o *cor.* 8. 6. Alſo it is manifeſt that BAq : ADq (*o*) : : BA : AC (*p*)
p *conſtr.* : : 3 : 2. *Which was to be demonſtrated.*

### *Coroll.*

1. ABq : HEq : : 9 : 2. For if ABq be put = 9,
q 12. 13. then ADq (EFq) ſhall be 9. (*q*) conſequently HEq ſhall be 2.

2. If L be the center, then ſhall AB : LC : : 6 :
r *conſtr.* 1. For if AB be put = 6. then AL ſhall be 3; (*r*) and thence AC 4. wherefore LC ſhall be 1. Hence,

3. AB : HI : : 6 : 4 : : 3 : 2. whence.

4. ABq : HIq : : 9 : 4.

## PROP. XIV. Plate VII. *Fig.* 7, 8.

*To deſcribe an Octaedron* KEFGDL, *and comprehend it in the given ſphere, wherein a pyramid is: and to demonſtrate that* AH, *the diameter of the ſphere, is in power double of* AC, *the ſide of that Octaedron.*

About AH deſcribe the ſemicircle ACH, and from the center B erect the perpendicular BC, draw AC, HC;
a 46. 1. then upon ED = AC (*a*) make the ſquare EFGD, whoſe diameters DF, EG, cut in the center I; from I, draw
b 12. 11. IL = AB (*b*) perpendicular to the plane EFGD, produce
c 3. 1. IL (*c*) till IK = IL. and join KE, KF, KG, KD, LE, LF, LG, LD; then ſhall KEFGDL be the Octaedron required.

For AB, BH, FI, IE, *&c.* being ſemidiameters of
d 4. 1. equal ſquares are equal one to the other; (*d*) whence the baſes LF, LE, FE, *&c.* of the right-angled triangles LIE, LIF, FIE, *&c.* are equal, and conſequently the eight triangles LFE, LFG, LGD, LDE, KEF, KFG, KGD,
e 27. *def.* 11. KDE, are equilateral, (*e*) and make an Octaedron, which may be inſcribed in a ſphere, whoſe center is I, and IL or AB the radius, (becauſe AB, IL, IF, IK, *&c.*
f *conſtr.* (*f*) are equal.) *Which was to be done.* Moreover, it
g 47. 1. is evident that AHq (LKq) (*g*) = 2 ACq (2 LDq.) *Which was to be demon.*

### *Coroll.*

1. Hence it is manifeſt, that in the Octaedron the three diameters EG, FD, LK cut one the other perpendicularly in the center of the ſphere.

2. Alſo, that the three planes EFGD, LEKG, LFKD are ſquares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal pyramids EFGDL, and EFGDK, whoſe common baſe is the ſquare EFGD.

4. Laſtly, it follows (*h*) that the oppoſite baſes of the h 15. 11.
Octaedron are parallel one to the other.

## PROP. XV. Plate VII. *Fig.* 9, 10.

*To deſcribe a cube* EFGHIKLM, *and comprehend it in the ſame ſphere, wherein the former figures were; and to demonſtrate that* AB *the diameter of the ſphere is in power triple to* EF, *the ſide of that cube.*

Upon AB deſcribe a ſemicircle ACB; (*a*) and make a 10. 6.
$AB = 3\,DA$, and from D raiſe the perpendicular DC,
and join BC and AC. Then upon $EF = AC$ (*b*) make b 46. 1.
the ſquare EFGH, upon whoſe plane let the right-lines
EI, FK, HM, GL, ſtand perpendicular, being equal to
EF, and connect them with the right-lines IK, KL, LM,
IM. The ſolid EFGHIKLM, is a cube, as is ſufficiently apparent from the conſtruction.

In the oppoſite ſquares EFKI, HGLM, draw the diameters EK, FI, HL, MG, through which let the planes
EKLH, FIMG be drawn, cutting one another in the line
NO, which (*c*) ſhall divide equally in two parts the diame- c *cor.* 39.
ters of the cube EL, FM, GI, HK, in P the center of the 11.
cube; (*d*) therefore P ſhall be the center of a ſphere paſſing d 15. *def.*
through the angular points of the cube. Moreover, 1. *and* 14.
$ELq\ (e) = EKq + KLq\ (e) = 3\,KLq$, (*f*) or 3 ACq. *def.* 11.
but $ABq : ACq\ (g) :: BA : DA\ (f) :: 3 : 1$; (*h*) there- e 47. 1.
fore $AB = EL$; wherefore we have made a cube, &c. f *conſtr.*
*Which was to be done.* g *cor.* 8. 6.
h 14. 5.

*Coroll.*

1. Hence it is manifeſt that all the diameters of the cube are equal one to the other, and do equally biſect one another in the center of the ſphere. And for the ſame reaſon the right-lines which conjoin the center of the oppoſite ſquares are biſected in the ſame center.

2. The diameter of a ſphere containeth in power the
ſide of a tetraedron and of a cube, *viz.* $ABq\ (k) = (l)$ k 47. 1.
$BCq + (m)\ ACq$. l 13. 13.
m 15. 13.

PROP.

## PROP. XVI. Plate VII. *Fig.* 11, 12.

*To deſcribe an Icoſaedron* ZGHIKFYVXRST, *and encompaſs it in the ſphere, wherein were contained, the foreſaid ſolids ; and to demonſtrate that* FG *the ſide of the Icoſaedron is that irrational-line, which is called a minor-line.*

Upon AB the diameter of a ſphere deſcribe the
a 10. 6. ſemicircle ADB ; and (*a*) make AB equal 5 BC, then
from C erect CD perpendicular, and draw AD and BD.
At the diſtance $EF = BD$ deſcribe the circle EFKNG ;
b 11. 4. (*b*) wherein inſcribe the equilateral pentagon FKIHG.
Divide equally in two parts the arches FG, GH,
&c. and join the right-lines FL, LG, &c. being the
c 12. 11. ſides of a decagon. Then (*c*) erect EQ, LR, MS, NT,
OV, PX equal to EF, and perpendicular to the plane
FKNG ; and connect RS, ST, TV, VX, XR ; as alſo
FX, FR, GR, GS, HS, HT, IT, IV, KV, KX.
Laſtly, produce EQ, and take $QY = FL$, and $EZ =$
FL, and conceive the right-lines ZG, ZH, ZI, ZK,
ZF to be drawn ; as alſo YV, YX, YR, YS, YT.
Then I ſay the Icoſaedron required is made.

d *conſtr.* For becauſe EQ, LR, MS, NT, OV, PX, are (*d*)
e 6. 11. equal and (*e*) parallel, alſo thoſe lines that join them
EL, QR, EM, QS, EN, QT, EO, QV, EP, QX,
f 33. 1. (*f*) are equal and parallel. And thence likewiſe LM
(or FG) RS, MN, ST, &c. are equal one to the
g 15. 11. other ; (*g*) therefore the plane drawn through EL,
EM, &c. is equidiſtant from the plane paſſing through
h 1. *def.* 3. QR, QS, &c. (*h*) and the circle QXRSTV drawn
from the center Q is equal to the circle EPLMNO,
and RSTVX is an equilateral pentagon. But EF, EG,
EH, &c. and QX, QR, QS, &c. being conceived to
k 47. 1. be drawn ; then becauſe $FRq\ (k) = FLq + LRq$, (*l*)
l *conſtr.* or $EFq\ (m) = FGq$. (*n*) therefore FR, FG, and ſo all
m 10. 13. RS, FG, FR, RG, GS, GH, &c. ſhall be equal one
n *ſch.* 48. 1. to the other, and conſequently the ten triangles RFX,
*and ax.* 1. KFG, RGS, &c. are equilateral and equal. Moreover,
o *cor.* 14. becauſe XQY is a (*o*) right-angle ; therefore
11. $XYq\ (p) = QXq + QYq\ (q) = VXq$ or FGq ; where-
p 47. 1. fore XY, VX, and likewiſe YV, YT, YS, YR, ZG,
q 10. 13. ZH, &c. are equal. Therefore other ten triangles are
made, equilateral and equal both to one another,
and to the ten former, and ſo an Icoſaedron is
made.

Moreover, divide equally EQ in *a*, draw the right
lines, *a* F, *a* X, *a* V; and becauſe QX (*r*) = QV, and r 15. *def.* 1.
*a* Q the common ſide, and EQX, EQV are right-an-
gles (*ſ*) therefore ſhall *a* X be = *a* V; and for the ſame ſ 4. 1.
reaſon all the lines *a* X, *a* R, *a* S, *a* T, *a* V, *a* F, *a* G,
*a* H, *a* I, *a* K are equal. But becauſe ZQ : QE (*t*) : : t 9. 13.
QE : ZE, therefore Z*a*q (*u*) = 5 E*a*q (*x*) = EQq (EFq) u 3. 13.
+ E*a*q (*y*) = *a*Fq; therefore Z*a* = *a*F; in like man- x 4. 2.
ner *a*F = Y*a*; therefore the ſphere, whoſe center is *a* y 47. 1.
and *a*F the radius, ſhall paſs through the 12 angular
points of the Icoſaedron.

Laſtly, (*z*) becauſe Z*a* : *a*E : : ZY : QE; (*a*) and ſo z 15. 5.
Z*a*q : *a*Eq : : ZYq : QEq. (*b*) therefore ZYq = 5 QEq, a 22. 6.
or 5 BDq; but ABq : BDq (*c*) : : AB : BC : : 5 : 1; (*d*) b 14. 5.
therefore ZY = AB, *Which was to be done.* Therefore c *cor.* 8. 6.
if AB be put ῥ, (*e*) then EF = √ AB × BC ſhall be d 1. *ax.* 1.
alſo ῥ, and conſequently FG the ſide of the pentagon, e *ſch.* 12.
and likewiſe of the Icoſaedron, (*f*) is a minor-line. 10.
*Which was to be dem.* f 11. 13.

*Coroll.*

1. From hence is inferred, that the diameter of the ſphere is in power quintuple of the ſemidiamater of the circle encompaſſing the five ſides of the Icoſaedron.

2. Alſo it is manifeſt that the diameter of the ſphere is compoſed of the ſide of a hexagon, that is, of the ſemidiameter, and two ſides of the decagon of a circle encompaſſing the five ſides of the Icoſaedron.

3. It appears likewiſe that the oppoſite ſides of an
Icoſaedron, ſuch as RX, HI, are parallels. For RX a 33. 1.
(*a*) is parallel to LP, (*b*) parallel to HI. b *ſch.* 26.
3.

### PROP. XVII. Plate VII. *Fig.* 13.

*To deſcribe a Dodecaedron, and comprehend it in the ſphere wherein the former figures were comprehended; and to demonſtrate that the ſide* RS *of the Dodecaedron is an irrational-line of that ſort which is called an apotome or reſidual-line.*

Let AB be a cube inſcribed in the given ſphere, and
let all the ſides thereof be divided equally in the points
E, H, F, G, K, L, &c. and join the right-lines KL, MH,
HG, EF; (*a*) make HI : IQ : : IQ : QH; and take NO, a 30. 6.
NP,

a 47. 1.
b 7. *ax.* 1.
c 4. 13.
d 47. 1.
e 4. 2.
f *conſtr.* 96. 11.
g 33. 1.
h 9. 1.
k 7. 11.
k *conſtr.*
l 6. 11.
m 32. 6.
n 1, *and* 2. 11.
o 5. 13.
p 47. 13.
q 1. *ax.* 6. *and* 4. 13.
r 4. 2.
ſ 8. 1.
* 7. 13.
t 15. 13.
u 1. *ax.* 1.
x 29. 1.
z 47. 1.
a 4. 13.
b 15. 13.

NP, = IQ; then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC; and let OR, PS, QT, be equal to IQ, NO, NP; whence DR, RS, SC, CT, TD, being connected, DRSCT ſhall be a pentagon of the Dodecaedron required. For draw NV parallel to OR, and having drawn NV out as far as the center of the cube X, join the right-lines DS, DO, DP, CR, CP, HV, HT, RX. Becauſe DOq (*a*) = DKq (*b*) (KNq)+KOq (*c*) = 3 ONq (3 ORq) (*d*) thence DRq = 4 ORq (*e*) = OPq, or, RSq. therefore DR = RS. By a like way of reaſoning DR, RS, SC, CT, TP are equal. But becauſe OR (*f*) is equal and (*g*) parallel to PS, therefore RS, OP, and conſequently RS, DC ſhall be alſo parallels; (*h*) therefore theſe with thoſe that conjoin them DR, CS, VH, are in one and the ſame plane. Moreover, becauſe HI : IQ (*k*) : : IQ (TQ) : QH (*k*) : : HN : NV; and both TQ, HN, and QH, NV (*k*) are perpendicular to the ſame plane, (*l*) and ſo alſo parallels, (*m*) THV ſhall be a right-line; (*n*) therefore the Trapezium DRSC, and the triangle DTS are in one plane extended through the right-lines DC, TV; (*o*) therefore DTCSR is a pentagon, and that alſo equilateral, by what is ſhewn already. Furthermore, becauſe PK : KN : : KN : NP; and DSq (*p*) = DPq + PSq (PNq) = (*p*) DKq + PKq + NPq, (*q*) thence DSq = DKq + 3 KNq = 4 DKq (4 DHq) (*r*) = DCq; therefore DS = DC; whence the triangles DRS, LCT, are equilateral one to another; (*ſ*) therefore the angle DRS = DTC, and likewiſe the angle CSR = DCT; therefore the * pentagon DTCSR is alſo equiangular. Moreover, becauſe AX, DX, CX, &c. are ſemidiameters of the cube (*t*) thence is XN = IH, or KN, (*u*) and ſo XV = KP; wherefore becauſe RVX, is a (*x*) right-angle, (*z*) thence RXq = XVq + RVq (NPq) = KPq + NPq (*a*) = 3 KNq (*b*) = AXq or DXq, &c therefore RX, AX, DX, and for the ſame reaſon XS, XT, AX, are equal one to another And if by the ſame method whereby the pentagon DTCSR was made, twelve like pentagons, touching the twelve ſides of the cube, be made, they ſhall compoſe a Dodecaedron; and a ſphere paſſing through their angular points, whoſe radius is AX, or RX, ſhall comprehend that Dodecaedron. *Which was to be done.*

Laſtly, becauſe KN : NO (*c*) : : NO : OK; (*d*) thence KL : OP : : OP : OK + PL. Therefore if AB the diameter

of the ſphere be ſuppoſed ῥ, then ſhall KL (*e*) $= \surd\frac{AB}{3}$ e 15. 3.
(*f*) be alſo ῥ. (*g*) whence OP, or RS the ſide of the Do- f *ſchol.* 12.
decaedron ſhall be a reſidual-line. *Which was to be dem.* 10.
g 6. 13.

*Coroll.*

From this demonſtration it follows, 1. That if the ſide of a cube be cut in extreme and mean proportion, the greater ſegment ſhall be the ſide of the Dodecaedron deſcribed in the ſame ſphere.

2. If the leſſer ſegment of a right-line, cut in extreme and mean proportion, be the ſide of the Dodecaedron, the greater ſegment ſhall be the ſide of the cube inſcribed in the ſame ſphere.

3. It is manifeſt alſo, that the ſide of the cube is equal to the right-line which ſubtendeth the angle of a pentagon of the Dodecaedron inſcribed in the ſame ſphere.

## PROP. XVIII. Plate VII. *Fig.* 14.

*To find out the ſides of the five precedent figures, and compare them together.*

Let AB be the diamter of the ſphere given, and AEB
the ſemicircle, and let AC be (*a*) $= \frac{1}{2}$ AB, and AD (*b*) a 10. 1.
$= \frac{1}{3}$ AB; then erect the perpendiculars CE, DF, and b 10. 6.
BG = AB; join AF, AE, BE, BF, CG; and let fall
the perpendicular HI from H, and CK being taken equal
to CI, from K erect the perpendicular KL, and join AL. c 30. 6.
Laſtly, (*c*) make AF : AO : : AO : OF. d *conſtr.*

Therefore 3 : 2 (*d*) : : AB : BD (*e*) : : ABq : BFq the e *cor.* 8. 6.
ſide of a Tetraedron, and 2 : 1 : : (*a*) AB : AC : : ABq : f 14. 13.
BEq (*f*) the ſide of an Octaedron. g 15. 13.

Alſo 3 : 1 (*d*) : : AB : AD (*e*) : : ABq : AFq. (*g*) the h *conſtr.*
ſide of an Hexaedron. k *cor.* 17.

Moreover, becauſe AF : AO (*h*) : : AO : OF; (*k*) 13.
thence ſhall AO be the ſide of a Dodecaedron. Laſtly, l 4. 6.
BG (2 BC) : BC (*l*) : : HI : IC; (*m*) therefore HI = 2 m 14. 5.
CI (*n*) = KI; therefore HIq (*o*) = 4 CIq. conſequently n *conſtr.*
CHq (*p*) = 5 CIq; (*q*) therefore ABq = 5 KIq. (*r*) o 4. 2.
therefore KI, or HI is a radius of a circle encloſing the p 47. 1.
pentagon of an Icoſaedron; and AK or IB (*r*) is the q 15. 5.
ſide of a decagon inſcribed in the ſame circle; (*ſ*) r *cor.* 16.
whence AL ſhall be the ſide of a pentagon, (*t*) and alſo 13.
the ſide of an Icoſaedron. Whereby it appears that BF, ſ 10. 13.
t 16. 13.

BE, AF are ῥητ. and AL, AO ῥητ, and BF ⊏ BE, and
BE ⊏ AF, and AF ⊏ AO. And becauſe 3 AFq =
u 1. 6. ABq (*u*) = 5 KLq, and AF × AO ⊏ AF × OF, (*x*) and
x 4. *ax.* 1. ſo AF × AO + AF × OF ⊏ 2 AF × OF, (*y*) that is,
y 1. 2. AFq ⊏ (*z*) 2 AOq; (*a*) thence ſhall 3 AFq (5 KLq)
z 17. 6. be ⊏ 6 AOq. conſequently KL ⊏ AO, and much ra-
a 47. 1. ther AL ⊏ AO.

That we may expreſs theſe ſides in numbers; If AB be ſuppoſed √ 60, then, reducing what is already ſhewn to ſupputation, BF = √ 40, and BE = √ 30, and AF = √ 20. Alſo AL = √ 30 — √ 180 (for AK = √ 15 -- √ 3. and KL (HI) = √ 12 :) Laſtly, AO = √ : 30 — √ 500 (√ 25 — √ 5.)

*Schol.*

*It is very apparent that beſides the five aforeſaid figures, there cannot be deſcribed any other regular ſolid figure* (viz. *ſuch as may be contained under ordinate and equal plane figures.*)

For three plane angles at leaſt are required to the con-
a 21. 11. ſtituting of a ſolid angle: (*a*) all which muſt be leſs than
b *See ſch.* four right-angles. (*b*) But 6 angles of an equilateral tri-
32. 1. angle, 4 of a ſquare, and 6 of a hexagon, ſeverally
equal 4 right-angles; and 4 of a pentagon, 3 of a heptagon, 3 of an octagon, *&c.* exceed 4 right-angles: Therefore only of 3, 4, or 5 equilateral triangles, of 3 ſquares, or 3 pentagons, it is poſſible to make a ſolid angle. Wherefore beſides the five above mentioned, there cannot be any other regular bodies.

## Out of P. Herigon.

*The Proportions of the ſphere and the five regular figures inſcribed in the ſame.*

Let the diameter of the ſphere be 2, then ſhall the periphery or circumference of the greater circle, be 6. 28318.

The ſuperficies of the greater circle, 3. 14159.

The ſuperficies of the ſphere, 12. 56637.

The ſolidity of the ſphere, 4. 1879.

The ſide of the tetraedron, 1. 62299.

The ſuperficies of the tetraedron, 4. 6188.

The ſolidity of the tetraedron, 0. 15132.

The ſide of the hexaedron, 1. 1547.
The ſuperficies of the hexaedron, 8.
The ſolidity of the hexaedron, 1. 5396.

The ſide of the octaedron, 1. 41421.
The ſuperficies of the octaedron, 6. 9282.
The ſolidity of the octaedron, 1. 33333.

The ſide of the dodecaedron, 0. 71364.
The ſuperficies of the dodecaedron, 10. 51462.
The ſolidity of the dodecaedron, 2. 78516.

The ſide of the Icoſaedron, 1. 05146.
The ſuperficies of the Icoſaedron, 9. 57454.
The ſolidity of the Icoſaedron, 2. 53615.

## *The End of the Thirteenth* BOOK.

# The FOURTEENTH BOOK OF EUCLIDE'S ELEMENTS.

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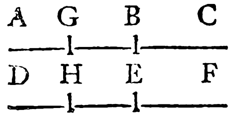
## PROPOSITION I. Plate VII. *Fig.* 15.

A *Perpendicular-line* DF *drawn from* D *the center of a circle* ABC *to* BC *the ſide of a pentagon inſcribed in the ſaid circle, is the half of theſe two lines taken together,* viz. *of the ſide of the hexagon* DE, *and the ſide of the decagon* EC *inſcribed in the ſame circle* ABC.

a 4. 1. Take FG = FE, and draw CG; (*a*) Then CE is =
b 5. 1. CG. Therefore the angle CGE (*b*) = CEG (*b*) = ECD;
c 32. 1. therefore the angle ECG (*c*) = EDC (*d*) = $\frac{1}{4}$ ADC (*e*)
d *hyp. and* = $\frac{1}{2}$ CED ($\frac{1}{2}$ ECD;) (*f*) conſequently the angle GCD
33. 6. = ECG = EDC, (*g*) wherefore DG = GC (CE.)
e 20. 3. f 7. *ax.* thereforc DF = CE (DG) + EF = $\frac{DE + CE.}{2}$ *Which*
g 6. 1. *was to be demonſtrated.*

## PROP. II.



*If two right-lines* AB, DE, *are cut according to extreme and mean proportion* (AB : AG : : AG : GB. *and* DE : DH : : DH : HE.) *they ſhall be cut after the ſame manner,* viz. *in the ſame proportions* (AG : GB : : DH : HE.)

Take BC = BG; and EF = EH. Then AB × BG is
a 17. 6. (*a*) AGq. wherefore ACq (*b*) = 4 ABG + AGq (*c*) = 5
b 8. 2. AGq. In like manner ſhall DFq = 5 DHq. (*d*) therefore
c 1. *ax.* 1. AC : AG : : DF : DH. whence by compounding AC +
d 22. 5. AG : AG : : DF + DH : DH. that is, 2 AB : AG : : 2
*and* 22. 6. DE : DH, (*e*) conſequently AB : AG : : DE : DH; (*f*)
e 22. 5. whence by diviſion AG : GB : : DH : HE. *Which was*
f 17. 5. *to be demonſtrated.*

PROP.

## PROP. III. Plate VII. *Fig.* 16, 17.

*The ſame circle* ABD *comprehends both* ABCDE *the pentagon of a Dodecaedron, and* LMN *the triangle of an Icoſaedron inſcribed in the ſame ſphere.*

Draw the diameter AG, and the right-lines AC, CG; and let IK be the diameter of the ſphere, (*a*) and IKq = 5 OPq. (*b*) and make OP : OQ : : OQ : QP. Becauſe ACq + CGq (*c*) = AGq (*d*) = 4 FGQ; and ABq (*e*) = FGq + CGq. (*f*) thence ACq + ABq = 5 FGq; moreover, becauſe CA : AB (*g*) : : AB : CA − AB; and OP : OQ : : OQ : QP; (*h*) and as CA : OP : : AB : OQ (*k*) therefore 3 ACq (*l*) (IKq.) : 5 OPq (*m*) (IKq) : : 3 ABq : 5 OQq. therefore 3 ABq = 5 OQq. But becauſe ML (*n*) is the ſide of a pentagon inſcribed in a circle, whoſe radius is OP, thence 15 RMq (*o*) = 5 MLq (*p*) = 5 OPq + 5 OQq = * 3 ACq + 3 ABq (*q*) = 15 FGq; (*r*) therefore RM = FG. (*ſ*) and conſequently the circle ABD is = the circle LMN. *Which was to be demonſtrated.*

a *ſch.* 47. 1.
b 30. 6.
c 47. 1.
d 4. 2.
e 10. 13.
f 2. *and* 3 *ax.*
g 8. 13.
h 2. 13. & 16. 5.
k 22. 6. & 4. 5.
l 15. 13.
m *conſtr.*
n *cor.* 16. 13.
o 12. 13.
p 10. 13.
q 15. 5. *and above*
* *before*
r 1. *ax.* 1. *and ſchol.* 48. 1.
ſ 1. *def.* 3.

## PROP. IV. *Fig.* 18, 19.

*If from* F *the center of a circle encompaſſing the pentagon of a Dodecaedron* ABCDE, *a perpendicular-line* FG *be drawn to one ſide of the Pentagon* CD; *the rectangle contained under the ſaid ſide* CD *and the perpendicular* FG, *being thirty times taken, is equal to the ſuperficies of a Dodecaedron. Alſo,*

*If from the center* L *of a circle incloſing the triangle of an Icoſaedron* HIK, *a perpendicular-line* LM *be drawn to one ſide of the triangle* HK, *the rectangle contained under the ſaid ſide* HK *and the perpendicular* LM, *being thirty times taken, ſhall be equal to the ſuperficies of an Icoſaedron.*

Draw FA, FB, FC, FD, FE; (*a*) then ſhall the trigles CFD, DFE, EFA, AFB, BFC be equal, but CD × FG (*b*) = 2 triangles CFD; therefore 30 CD × GF (*c*) = 60 CFD (*d*) = 12 pentagons ABCDE (*e*) = to the ſuperficies of a Dodecaedron. *Which was to be demonſtrated.*

Draw LI, LH, LK; then HK × LM (*f*) = 2 triangles LHK; therefore 30 HK × LM (*g*) = 60 HLK = 20 HIK (*h*) = the ſuperficies of an Icoſaedron. *Which was to be demonſtrated.*

a 8. 1.
b 41. 1.
c 15. 5.
d 6. *ax.*
e 17. 3.
f 41. 1.
g 15. 5.
h 16. 13.

### *Coroll.*

k 15. 5. CD × FG : HK × LM (*k*) : : the ſuperficies of a Dodecaedron to the ſuperficies of an Icoſaedron.

### PROP. V. Plate VII. *Fig.* 20.

*The ſuperficies of a Dodecaedron hath to the ſuperficies of an Icoſaedron inſcribed in the ſame ſphere, the ſame proportion that* H *the ſide of a cube hath to* AD *the ſide of an Icoſaedron.*

a 3. 14. Let the circle ABCD (*a*) encloſe both the Pentagon of
a Dodecaedron, and the triangle of an Icoſaedron; whoſe
ſides are BD, AD; upon which from the center E let
fall the perpendiculars EF, EGC; and draw CD.
b 9. 13. Becauſe EC+CD : EC (*b*) : : EC : CD. thence EG
c 1, 14. (*c*) ($\frac{1}{2}$ EC × $\frac{1}{2}$ CD) : EF (*d*) ($\frac{1}{2}$ EC) (*e*) : : EF : EG—EF
d *cor.* 12, ($\frac{1}{2}$ CD.) but H : BD (*f*) : : BD : H—BD; (*g*) therefore
13. H : BD : : EG : EF. conſequently H × EF = BD × EG.
e 15. 5. wherefore ſince H : AD (*h*) : : H × EF : AD × FF; (*k*)
f *cor.* 17, thence ſhall be H : AD : : BD × EG : AD × EF (*l*); :
13. the ſuperficies of a Dodecaedron to the ſuperficies of an
g 2. 14. Icoſaedron. *Which was to be demonſtrated.*
h 1. 6.
k 7. 5.
l *cor.* 4. 14.

### PROP. VI. *Fig.* 21.

*If a right-line* AB *be cut in extreme and mean proportion, then as the right-line* BF, *containing in power that which is made of the whole line* AB, *and that which is made of the greater ſegment* AC, *is to the right-line* E *containing in power that which is made of the whole line* AB, *and that which is made of the leſſer ſegment* BC; *ſo is the ſide of the cube* BG *to the ſide of an Icoſaedron* BK *inſcribed in the ſame ſphere with the cube.*

In the circle, whoſe ſemidiameter is AB, inſcribe
BFGHI the pentagon of a Dodecaedron; and BKL the
a *cor.* 17. triangle of an Icoſaedron, (*a*) wherefore BG ſhall be the
13. ſide of a cube inſcribed in the ſame ſphere; therefore
b 12. 13. BKq (*b*) = 3 ABq; and Eq (*c*) = 3 ACq. therefore BKq :
c 4. 13. Eq (*d*) : : ABq : ACq (*e*) : : BGq : BFq; wherefore by
d 15. 5. permution BGq : BKq : : BFq : Eq; (*f*) whence BG :
e 2. 14. BK : : BF : E. *Which was to be demonſtrated.*
f 22. 6.

PROP.

## PROP. VII.

*A Dodecaedron is to an Icoſaedron, as the ſide of a Cube is to the ſide of an Icoſaedron, inſcribed in one and the ſame ſphere.*

Becauſe (*a*) the ſame circle comprehends both the pen- a 3. 14.
tagon of a Dodecaedron, and the triangle of an Icoſae-
dron, (*b*) the perpendiculars drawn from the center of the b 47. 1.
ſphere to the planes of the pentagon and triangle, ſhall
be equal one to another. Therefore if the Dodecaedron
end Icoſaedron be conceived and divided into pyramids,
right-lines being drawn from the center of the ſphere
to all the angles, the altitudes of all the pyramids ſhall
be equal one to the other. Wherefore ſince the pyra-
mids are (*c*) of equal height with the baſes, and the ſu- c 5. *and* 6.
perficies of the Dodecaedron is equal to twelve penta- 12.
gons, and the ſuperficies of the Icoſaedron to twenty
triangles, the Dodecaedron ſhall be to the Icoſaedron, as
the ſuperficies of the Dodecaedron is to the ſuperficies of
the Icoſaedron; (*d*) that is, as the ſide of the cube is to d 5. 14.
the ſide of the Icoſaedron.

## PROP. VIII. Plate VII. *Fig.* 22, 23.

*The ſame circle* BCDE *comprehends both the ſquare of the cube* BCDE; *and the triangle of the octaedron* FGH *inſcribed in one and the ſame ſphere.*

Let A be the diameter of the ſphere. Becauſe Aq (*a*) a 15. 13.
= 3 BCq (*b*) = 6 BIq; and alſo Aq (*c*) = 2 GFq (*d*) = b 47. 1.
6 KFq; thence ſhall BI = KF; (*e*) therefore the circle c 14. 13.
CBED=GFH. *Which was to be demonſtrated.* d 12. 13.
e 1. *def.* 3.

### *The End of the Fourteenth* BOOK.

# The FIFTEENTH BOOK

# OF

# EUCLIDE's

# ELEMENTS.

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### RROPOSITION I. Plate VII. *Fig.* 24.

I*N a cube given* ABGHDCFE *to deſcribe a pyramid* AGEC.

From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA.
a 47. 1. All which are (*a*) equal among themſelves, as being the diameters of equal ſquares: therefore the triangles CAG, CGE, CEA, EAG are equilateral and equal; and conſequently AGEC is a pyramid, which inſiſts upon the
b 31. *def.* 11. angles of the cube, and therefore (*b*) is inſcribed in it. *Which was to be done.*

### PROP. II. *Fig.* 25.

*In a pyramid given* ABDC *to deſcribe an Octaedron* EGKIFH.

a 10. 1. (*a*) Biſect the ſides of the pyramid in the points E, I, F, K, G, H, which join with the 12 right-lines EF,
b 4. 1. FG, GE, &c. All theſe are (*b*) equal one to the other; conſequently the 8 triangles EHI, IHK, &c. are equi-
c 27. *def.* 11. lateral and equal, and ſo make (*c*) an Octaedron deſcri-
d 31. *def.* 11. bed (*d*) in the given pyramid. *Which was to be done.*

## PROP. III. Plate VII. *Fig.* 26.

*In a cube given* CHGBDEFA *to deſcribe an Octaedron* NPQSOR.

Connect * the centers of the ſquares N, P, Q, S, O, R * 8, 4.
with the twelve right-lines NP, PQ, QS, *&c.* which
are (*a*) equal among themſelves; and ſo make 8 equilateral a 4. 1.
and equal triangles: wherefore (*b*) the Octaedron b 31. *and*
NPQSOR (*b*) is inſcribed in the cube. *Which was to be* 27. *def.* 11.
*done.*

## PROP. IV. *Fig.* 27.

*In an Octaedron given* ABCDEF, *to inſcribe a cube.*

Let the ſides of the pyramid EABCD, whoſe baſe
is the ſquare ABCD, be biſected by the right-lines,
LM, MN, NO, OL, which are (*a*) equal and (*b*) parallel a 4. 1.
to the ſides of the ſquare ABCD; (*c*) then b 2. 6.
the quadrilateral LMNO is a ſquare. In like manner, c 29. *def.* 1.
if the ſides of the ſquare LMNO be biſected in the
points G, H, K, I, and GH, HK, KI, IG connected,
GHKI ſhall be a ſquare. And if in the other 5
pyramids of the Octaedron, the centers of the triangles
be in the ſame ſort conjoined with right-lines, then
other ſquares will be deſcribed like and equal to the
ſquare GHKI; wherefore ſix ſuch ſquares ſhall make
a cube, which ſhall be deſcribed within an Octaedron,
(*d*) ſince its eight angles touch the eight baſes of the d 31. *def.*
Octaedron in their centers. *Which was to be done.* 11.

## PROP. V. *Fig.* 28.

*In an Icoſaedron given to inſcribe a Dodecaedron.*

Let ABCDEF be a pyramid of the Icoſaedron, whoſe baſe is the pentagon ABCDE; and the centers of the triangles G, H, I, K, L; which connect with the right-lines GH, HI, IK, KL, LG. Then GHIKL ſhall be a pentagon of the Dodecaedron to be inſcribed.

For the right-lines, FM, FN, FO, FP, FQ, paſſing
through the centers of the triangles, (*a*) biſect their baſes; a *cor.* 3. 3.
(*b*) therefore the right-lines MN, NO, OP, PQ, QM, b 4. 1.
(*c*) are equal one to the other; (*d*) whence alſo the angles c 4. 1.
MFN d 8. 1.

MFN, NFO, OFP, PFQ, QFM are equal; therefore the pentagon GHIKL is equiangular,(*e*) and conſequently equilateral, ſince FG, FH, FI, FK, FL (*f*) are equal. And if in the other eleven pyramids of the Iſocaedron, the centers of the triangles be in like ſort joined with right-lines, then will the pentagons be equal and like to the pentagon GHIKL be deſcribed. Wherefore 12 of ſuch pentagons ſhall conſtitute a Dodecaedron; which alſo ſhall be deſcribed in the Icoſaedron, ſeeing the twenty angles of the Dodecaedron conſiſt upon the centers of the twenty baſes of the Icoſaedron. Whereby it appears that we have deſcribed a Dodecaedron in an Icoſaedron given. *Which was to be done.*

e 4. 1.
f 12. 13.

*FINIS.*

# *EUCLIDE's DATA.*

## DEFINITIONS.

I. *PLanes or Spaces, Lines and Angels, to which we can find others equal, are ſaid to be given in Magnitude.*

II. *A Ratio is ſaid to be given, when we can find it, or one equal to it.*

III. *Rectiline-figures, whoſe angles are given, and alſo the ratio of the ſides to one another, are ſaid to be given in Species or Kind.*

IV. *Points, Lines and Angles, which have and keep always one and the ſame place and ſituation, are ſaid to be given in Poſition or Situation.*

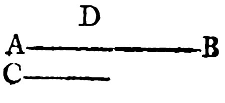
V. *A Circle is ſaid to be given in Magnitude, when the ſemidiameter thereof is given in Magnitude.*

VI. *A circle is ſaid to be given in Poſition, and Magnitude, when the center thereof is given in Poſition, and the ſemidiameter in Magnitude.*

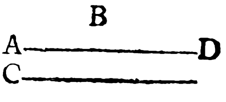
VII. *Segments of Circles, whoſe angles and baſes are given in Magnitude, are ſaid to be given in Magnitude.*

VIII. *Segments of a Circle, whoſe angles are given in Magnitude, and the baſes of the ſegments in Poſition and Magnitude, are ſaid to be given in Poſition and Magnitude.*

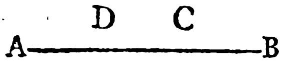
IX. *A Magnitude* AB, *is greater than another Magnitude* C, *by a given Magnitude* BD, *when having taken away the given Magnitude* DB, *the reſt* AD, *is equal to the other Magnitude* C.



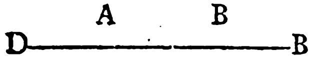
X. *A Magnitude* AB, *is leſs than another Magnitude* C, *by a given Magnitude* BD, *when having added thereto the given Magnitude* BD, *the whole* AD *is equal to the other Magnitude* C.



XI. *A Magnitude* AB *is ſaid it be greater than another Magnitude* CB, *by a given Magnitude* AD, *and in ratio, when taken from the ſame Magnitude the given Magnitude* AD, *the reſt* DB, *hath to the other Magnitude* CB, *a given ratio.*



XII. *A Magnitude* AB *is ſaid to be leſs than another Magnitude* BC, *by a given Magnitude* AD, *and in ratio, when the given magnitude* AD *being added thereto, the whole* DB *hath to the other Magnitude* BC, *a given ratio.*



XIII. *A right-line is ſaid to be drawn down from a given point, unto a right-line given in Poſition, the right-line being drawn in a given angle.*

XIV. *A right-line is ſaid to be drawn up from a given point, to a right-line given in Poſition, the right-line being drawn in a given-angle.*

XV. *A right-line is againſt another right-line in Poſition, when it is drawn parallel thereto through a given point.*

## PROPOSITION I.

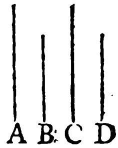
*TWO Magnitudes* A *and* B *being given, the ratio they have to one another* A *to* B *is alſo given.*

a 1. *def.* *Demonſtration.* For ſeeing that the Magnitude A is given, (*a*) we can find one equal thereto, which let be C. Again, foraſmuch as the Magnitude B is given, we can alſo find one equal to that, and let that be D. Therefore ſeeing
b 7. 5. that A is equal to C, and B to D, as A is to C, (*b*)
c 16. 5. ſo is B to D, and by permutation, (*c*) as A ſhall be to
d 2. *def.* B, ſo C ſhall be to D. Therefore (*d*) the ratio of A to B is given, for it is the ſame ratio as of C to D, as we have found, and which ought to be demonſtrated.

## PROP. II.

*If a given Magnitude* A, *hath to ſome other Magnitude* B, *a given ratio, that other Magnitude* B, *is alſo given in Magnitude.*

*Demon.*

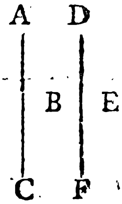


*Demonſt.* For ſeeing that A is given, we can find one equal thereto, which let be C: And foraſmuch as the ratio of A to B, is alſo given, we can find (*a*) one of the ſame. Let it be found, and let the ratio be of C to D. Now ſeeing that as A is to B, ſo C is to D; and by permutation, as A is to C, ſo B is to D: But A is equal to C, therefore (*b*) B ſhall be alſo equal to D. Therefore (*c*) the Magnitude B is given, ſeeing that thereto there hath been found one equal, to wit, D.

a 2. *def.*
b 14. 5.
c 1. *def.*

## PROP. III.

*If given magnitudes* AB *and* BC, *are compounded, that magnitude* AC, *that is compounded of them, ſhall be alſo given.*

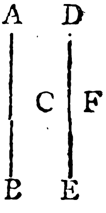


*Demonſtr.* For ſeeing that AB is given, we can find one equal to it, which let be DE. Again, ſeeing that BC is given we can alſo find one equal to that, which let be EF. Wherefore ſeeing that DE is equal to AB, and EF is equal to BC, the whole AC (*a*) is equal to the whole DF. Therefore AC is given, ſeeing that DF is propoſed equal thereto.

a 2. *ax.* 1.

## PROP. IV.

*If from a given magnitude* AB, *there be taken away a given magnitude* AC, *the remaining magnitude* CB *is alſo given.*

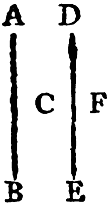


*Demonſtr.* Foraſmuch as AB is given, we can find one equal thereto, which let be DE. Again, ſeeing that AC is given, we can alſo find one equal to it, which let be DF. Seeing then that the Magnitude AB is equal to the Magnitude DE, and the Magnitude AC to the Magnitude DF; the reſt CB (*a*) ſhall be equal to the reſt FE. Wherefore CB is given, for to it there hath been found an equal, to wit, FE.

a 3. *ax.* 1.

PROP.

## PROP. V.



*If a Magnitude* AB, *hath a given ratio to ſome part thereof* AC, *it will have alſo a given ratio to the part remaining* CB.

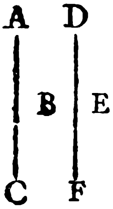
*Demonſtr.* Let DE be expoſed as a given Magnitude, and ſeeing that the ratio of the Magnitude AB, to Magnitude AC, is given (*a*) we can find one of the ſame, which let be DE to DF; therefore the ratio of the ſame DE to DF is given. But DE being given, ſo is (*b*) alſo its part DF; and conſequently, (*c*) the reſt FE: Therefore (*d*) ſeeing that DE and FE are given, the ratio of the ſame DE to FE is alſo given. And foraſmuch as DE is (*e*) to DF, as AB is to AC, and by converſion, as DE to FE, ſo AB is to CB. But the ratio of DE to FE, is given, as hath been demonſtrated; therefore the ratio of AB to CB is alſo given.

a 2. *def.*
b 2. *prop.*
c 4. *prop.*
d 1. *prop.*
e 19. 5.

### *Scholium.*

From this it is evident that if a Magnitude hath to ſome part thereof a given ratio, by diviſion, the ratio that one part hath to the other, ſhall be alſo given. For ſeeing that as DE is to FE, ſo is AB to CB; by diviſion, as DF to FE, ſo AC to CB. But it hath been demonſtrated that the parts DF and EF are given, and conſequently their ratio is alſo given: In like manner, therefore, the ratio of AC to CB is given.

## PROP. VI.



*If two Magnitudes* AB *and* BC, *having to one another a given ratio, are compounded, the Magnitude* AC *compounded of them, ſhall alſo have a given ratio to each of them* AB *and* BC.

*Demonſtr.* Let the given Magnitude DE be propoſed, and ſeeing that the reaſon of AB to BC is given; let there be made one and the ſame of the ſaid DE to EF; therefore the ratio of the ſame DE to EF is given; and therefore (*a*) the Magnitude DE being given, both the one and the other of them DE and FE, is given. Wherefore (*b*) the whole DF ſhall be alſo given. Therefore (*c*) the raito of the ſame DF to each of them DE and FF, ſhall be given. And

a 2. *prop.*
b 3. *prop.*
c 1. *prop.*

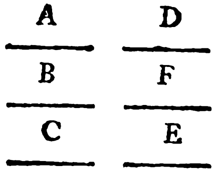
And foraſmuch then as AB is to BC, ſo is DE to EF;
by compounding, (*d*) as AC is to BC, ſo is DF to EF: d 18. 5.
Therefore by converſion, as AC to AB, ſo is DF to
DE. Therefore as the whole DF is to each of the other
Magnitudes DE and FE, ſo the whole AC is to each of
the Magnitudes AB and BC. Therefore (*e*) the ratio of e 2 *def.*
the ſame AC to each of the Magnitudes AB and BC is
given.

## PROP. VII.

*If a given Magnitude* AB *be divided according to a given ratio* AC *to* CB, *each ſegment* AC *and* CB *is given.*

*Demonſtr.* For ſeeing the ratio of AC to CB is given,
the ratio of (*a*) AB to each of them (AC and CB) is a 6. *prop.*
alſo given. But AB is given: Therefore (*b*) each of the b 2. *prop.*
ſegments AC and CB is alſo given.

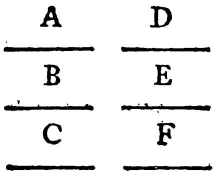
## PROP. VIII.



*Magnitudes* A *and* C, *which have to one and the ſame a given ratio,* B, *ſhall be to one another in a given ratio* A *to* C.

*Demonſtr.* For let the given magnitude D be propoſed, and ſeeing that the ratio of A to B is given,
let the ſame be done of the ſaid D to E. Now ſeeing
that D is given, (*a*) E is alſo given. Again, ſeeing that a 2. *prop.*
the ratio of B to C is given, let the ſame be done of
E to F. But E is given, and therefore F is alſo given.
But ſeeing that D is given, (*b*) the ratio of the ſame D b 1. *prop.*
to F is given; and ſeeing that as A to B, ſo is D to E,
and as B to C, ſo is E to F; in ratio of equality, (*c*) c 22 5.
as A is to C, ſo is D to F; but the ratio of D to F
is given. Therefore the ratio of A to C is alſo
given.

## PROP. IX.



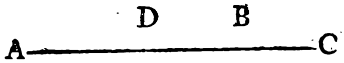
*If two or more magnitudes* A, B, *and* C, *are to one another in a given ratio, and that the ſame magnitudes* A, B, *and* C, *have to other magnitudes* D, E, *and* F, *given ratio's, although they be not the ſame, thoſe other magnitudes* D, E, *and* F *ſhall be alſo to one another in given ratio's.*

*Demonſtr.* Foraſmuch as the ratio of A to B is given, as alſo that of A to D, the ratio of D to B ſhall be given: But the ratio of B to E is alſo given; therefore the ratio of the ſame D to E ſhall be in like manner given. Again, ſeeing that the ratio of B to C is given, and alſo that of B to E, the ratio of E to C ſhall be given. But the ratio of C to F is alſo given. Therefore (*a*) the ratio of E to F ſhall be given. But it hath been demonſtrated that the ratio of D to E is alſo given; and therefore (*b*) the ratio of D to F ſhall be given. Therefore the magnitudes D, E, and F are to one another in given ratio's.

a 8. *prop.*

b 8. *prop.*

## PROP. X.



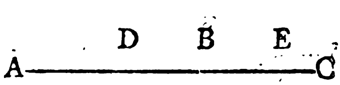
*If a magnitude* A B, *be greater than another magnitude* BC, *by a given magnitude, and in ratio, the magnitude* AC *compounded of both, ſhall be alſo greater than that the ſame magnitude, by a given magnitude, and in ratio; but if that compounded magnitude be greater than the ſame magnitude, by a given magnitude, and in ratio; either the remainder ſhall be alſo greater than that ſame, by a given magnitude, and in ratio; or elſe the ſame remainder is given with the following, to which the other magnitudes hath a given ratio.*

*Demonſtr.* For ſeeing that AB is greater than BC by a given magnitude, and in ratio, let the given magnitude AD be taken away. Therefore (*a*) the reaſon of the remainder DB to BC is given; and by compounding, (*b*) the ratio of DC to BC is alſo given. But the magnitude AD is alſo given; therefore AC is greater than the ſame BC by a given magnitude, and in ratio.

a 11. *def.*

b 6. *prop.*

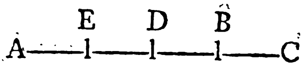
Again,



Again, Let the magnitude AC be greater than the magnitude BC, by a given magnitude, and in ratio; I ſay, that the reſt AB, is either greater than the ſame BC by a given magnitude, and in ratio; or that the ſame AB, with that which followeth, to which BC hath a given ratio, is given.

Foraſmuch as the magnitude AC is greater than the magnitude BC, by a given magnitude, and in ratio, cut off from it the given magnitude: Now the ſame given magnitude is either leſs than the magnitude AB, or greater: Let it in the firſt place be leſs, and let it be AD. Therefore the ratio of the remainder DC to CB is given. Wherefore by diviſion, the ratio of DB to BC is given. But the magnitude AD is alſo given; therefore the magnitude AB is greater (*c*) than the magnitude BC by a given magnitude, and in ratio. c 11. *def.* Now let the given magnitude be greater than the magnitude AB, and let AE be put equal thereto; therefore (*d*) the ratio of the remainder EC to CB is given; and d 11. *def.* by converſion, (*e*) the ratio of the ſame BC to BE, is e 5. *prop.* alſo given. But the ſame EB with BA is given, for that the whole AE is given: Therefore there is given AB, with that which follows BE, to which BC hath a given ratio.

## PROP. XI.



*If a magnitude* AB *be greater than a magnitude* BC, *by a given magnitude, and in ratio, the ſame magnitude* AB, *ſhall be alſo greater than the magnitude compounded of them by a given magnitude; and in ratio, and if the ſame magnitude be greater than the two others together by a given magnitude, and in ratio; that ſame magnitude ſhall be alſo greater than the reſt by a given magnitude, and in ratio.*

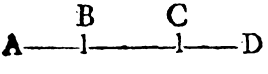
*Demonſtr.* For ſeeing that the magnitude AB is greater than BC by a given magnitude and in ratio; let there be taken from it a given magnitude AD: Therefore (*a*) the ratio of the reſt DB to DC, is given, and a 11. *def.* therefore (*b*) the ratio of DC to BD ſhall be alſo given: b 6. *prop.* Let the ſame be done of AD to DE, therefore the ratio of the ſame AD to DE is given. But AD is given, c 2. *prop.* therefore (*c*) DE is alſo given, and conſequently, (*d*) the d 4. *prop.*

reſt AE, is alſo given. But ſeeing that as AD : DE : :
e 16. 5. DC : BD; by permutation, (*e*) as AD : DC : : DE :
f 18. 5. DB : Therefore by compounding, (*f*) as AC : CD : :
g 16. 5. EB : DB; and by permutation, (*g*) as AC : EB : : DC :
DB. But the ratio of DC to DB is given: Therefore
alſo is AC to EB, and conſequently that of EB to AC.
But it hath been demonſtrated that AE is given, there-
h 11. *def.* fore (*h*) AB is greater than AC by a given Magnitude,
and in ratio.

But now let AB be greater than AC by a given Magnitude, and in ratio: I ſay, that the ſame AB is alſo greater than the reſt BC by a given Magnitude, and in ratio.

For ſeeing that AB is greater than AC by a given
Magnitude, and in ratio. Let the given Magnitude
i 11. *def.* AB be cut off therefrom: Therefore (*i*) the ratio of the
remainder EB to AC is given, and conſequently alſo
ſhall be given that of AC to EB. Let the ſame be done
of AD to DE, therefore the ratio of AD to DE is gi-
k 5. *prop.* ven; and by converſion, (*k*) the ratio of AD to AE ſhall
be alſo given, and conſequently that of AE to AD. Now
l 2. *prop.* AE is given, therefore the whole AD (*l*) ſhall be alſo
given; and ſeeing that as the whole AC is to the whole
EB, ſo the part cut off AD, is to the part cut off ED,
m 19. 5. ſo alſo ſhall be (*m*) the remainder DC to the remainder
DB. But the ratio of AC to EB is given: Therefore
alſo ſhall be given that of DC to DB. Wherefore by
n *ſchol.* 5. *prop.* diviſion (*n*) the ratio of BC to DB is given; and con-
ſequently alſo ſhall be given that of DB to BC. But it
o 11. *def.* hath been demonſtrated that AD is given: Therefore (*o*)
AB is greater than the ſame BC by a given Magnitude,
and in ratio.

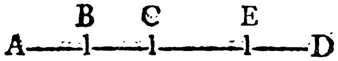
## PROP. XII.



*If there are three magnitudes* AB, BC, *and* CD, *and that the firſt* AB, *with the ſecond* BC, *to wit,* AC, *be given. And the ſecond* BC, *with third* CD, *to wit, BD, be alſo given: Either the firſt* AB *ſhall be equal to the third* CD, *or the one ſhall be greater than the other by a given Magnitude.*

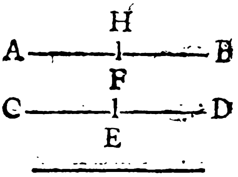
*Demonſtr.* Foraſmuch as each of the Magnitudes AC and BD are given, the given Magnitudes are either equal to one another, or unequal. Let them be firſt equal: Therefore AC is equal to BD; take away the common

Magnitude BC, and there will remain (*a*) AB, equal a 3. *ax.* 1.
to CD. But ſuppoſe them to be unequal as in this ſecond figure, and let BD be greater than AC: Let then BE be put equal to AC.



Now ſeeing that AC is given, BE is alſo given. But the whole BD is alſo given,
the reſt ED (*b*) ſhall be ſo alſo; and foraſmuch as BE is b 4. *prop.*
equal to AC, taking away the common Magnitude (*c*) c 3. *ax.* 1.
BC, there will remain AB equal to CE: But ED is given; Therefore CD is greater than AB by the given Magnitude ED.

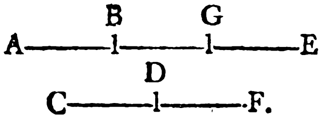
## PROP. XIII.



*If there are three magnitudes* AB, CD, *and* E, *and that the firſt of them* AB, *hath a given ratio to the ſecond* CD; *but the ſecond* CD *is greater than the third* E, *by a given magnitude, and in ratio; alſo the firſt* AB, *ſhall be greater than the third* E, *by a given magnitude, and in ratio.*

*Demonſtr.* For ſeeing that CD is greater than E by a given Magnitude, and in ratio; let the given Magnitude CF be taken therefrom: Therefore the ratio of the reſt FD to E is given. And foraſmuch as the ratio of AB to CD is given, let the ſame be done of AH to CF. Therefore the ratio of the ſame AH to CF is
given. But CF is given: Therefore (*a*) AH is alſo given. a 2. *prop.*
And ſeeing that as the whole AB is to the whole CD,
ſo is the part cut off AH is to the part cut off CF, and ſo
(*b*) alſo the reſt HB is to the reſt FD, the ratio of the b 19. 5.
ſame HB to FD is alſo given. But the ratio of FD to
E is alſo given: Therefore (*c*) the ratio of HB to F is c 8. *prop.*
given. But it hath been demonſtrated that AH is given: Therefore (*d*) AB is greater than the ſaid E by a d 11. *def.*
given Magnitude, and in ratio.

## PROP. XIV.

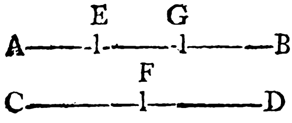


*If two Magnitudes* AB *and* CD, *have to one another a given ratio, and that to each of them there be added a given Magnitude, to wit,* BE *and* DF; *either the whole* AE *and* CF *ſhall have to one another a given ratio, or the one ſhall be greater than the other by a given Magnitude, and in ratio.*

*Demonſtr.* For ſeeing that each of thoſe Magnitudes a 1. *prop.* BE and DF, is given, *(a)* the ratio of the ſaid BE and DF is alſo given; and if that ratio be the ſame with that of AB to CD, that of the whole AE to the b 12. 5. whole CF, (*b*) ſhall be the ſame; and therefore the ratio of the ſaid AE to CF is given.

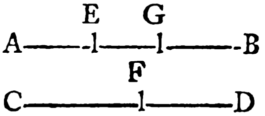
Now let the ratio of BE to DF be not the ſame with that of AB to CD, and let it be as AB to CD, ſo is BG to DF. Therefore the ratio of the ſaid BG to DF is given. But the Magnitude DF is given, there- c 2. *prop.* fore (*c*) BG is alſo given; and ſeeing that the whole BE d 4. *prop.* is given, (*d*) the reſt GE ſhall be alſo given. But for- e 12. 5. aſmuch as AB is to CD, as BG is to DF, (*e*) ſo alſo is the whole AG to the whole CF; and therefore the ratio of the ſaid AG to CF is given: But the Magnitude GE is f 11. *def.* given: Therefore *(f)* the Magnitude AE is greater than the Magnitude CF by a given Magnitude, and in ratio.

## PROP. XV.

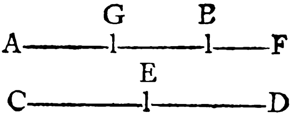


*If two Magnitudes* AB *and* CD, *have to one another a given ratio, and that from each of them be taken away a given Magnitude (to wit, from the Magnitudt* AB *the Magnitude* AE, *and from the Magnitude* CD *the Magnitude* CF) *the remaining Magnitudes* EB *and* FD, *either ſhall have to one another, a given ratio, or the one of them ſhall be greater than the other by a given Magnitude, and in ratio.*

*Demonſtr.* For ſeeing that each Magnitude AE and CF is given, the ratio of AE to CF is given; and if it be the ſame with that of AB to CD, that of the remainder EB to the remainder FD, (*a*) ſhall be alſo the ſame; and a 19. 5. therefore the ratio of the ſaid EB to FD ſhall be alſo given. But if it be not the ſame, let it be as AB to CD, ſo is AC to CF. Now the ratio of AB to CD is given, therefore alſo that of AG to CF ſhall be given. But CF is given, therefore (*b*) AG is given. But AE is alſo given, therefore (*c*) b 2. *prop.* the reſt EG is given; and ſeeing that as AB is to CD, c 4. *prop.* ſo the part cut off AG is to the part cut off CF, and ſo alſo is (*d*) the reſt GB to the reſt FD; the ratio of the d 4. *prop.* ſaid GB to FD is alſo given. Therefore ſeeing that EG e 11. *def.* is given, EB is greater than FD (*e*) by a given Magnitude, and in ratio.



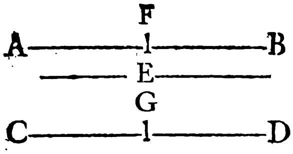
## PROP. XVI.



*If two Magnitudes* AB *and* CD, *have to one another a given ratio, and that from one of them, to wit,* CD, *there be taken away a given Magnitude* DE, *and to the other* AB *there be added a given Magnitude* BF, *the whole* AF *ſhall be greater than the reſt* CE, *by a given Magnitude, and in ratio.*

*Demonſtr.* For ſeeing that the ratio of AB to CD is given, let the ſame be made of BG to DE: Therefore (*a*) the ratio of the ſaid BG to DE is given. But DE is a 2. *def.* given, therefore (*b*) BG is alſo given. But BF is alſo b 2. *prop.* given, therefore (*c*) the whole GF is given. And ſeeing c 3. *prop.* that as AB is to CD, ſo the part cut off BG, is to the part cut off DE; and (*d*) ſo alſo is the remainder AG to the d 19. 5. remainder CE; the ratio of the ſaid AG to CE is given, But GF is given, therefore the Magnitude AF is greater than the Magnitude CE by by a given Magnitude, and in ratio.

## RPOP. XVII.

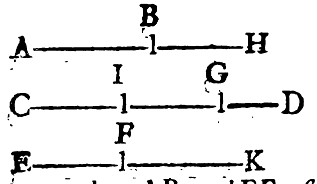


*If there are three Magnitudes AB, E, and CD, and that the firſt AB be greater than the ſecond E by a given Magnitude, and in ratio: And the third CD be alſo greater than the ſame ſecond E, by a given Magnitude, and in ratio; the firſt AB ſhall have to the third CD, either a given ratio, or elſe the one ſhall be greater than the other by a given Magnitude, and in ratio.*

*Demonſtr.* For ſeeing that AB is greater than E by a given Magnitude, and in ratio, let the Magnitude AF be taken away: Therefore the ratio of the remainder FB to E is given. Again, ſeeing that CD is greater than the ſaid E by a given Magnitude, and in ratio, let the given Magnitude CG be cut off therefrom; and the ratio of the remainder GD to E ſhall be given: Therefore (*a*) the ratio of FB to GD ſhall be alſo given. But to the ſaid FB and GD are added the given Magnitudes AF and CG: Therefore the whole AB and CD (*b*) ſhall either have to one another a given ratio, or the one ſhall be greater than the other by a given Magnitude, and in ratio.

a 8. *prop.*
b 14. *prop.*

## PROP. XVIII.



*If there are three Magnitudes AB, CD, and EF, and that the one of them, to wit, CD, be greater than either of the other AB or EF, by a given Magnitude, and in ratio; either the two others AB and EF, ſhall have to one another a given ratio, or the one ſhall be greater than the other by a given Magnitude, and in ratio.*

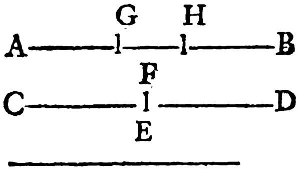
*Demonſtr.* Foraſmuch as the Magnitude CD is greater than the Magnitude AB by a given Magnitude, and in ratio, let the given Magnitude LG be taken therefrom: Therefore the ratio of the remainder CG to AB is given. Let the ſame be made of GD to BH, therefore the ratio of the ſaid DG to BH is given. But DG is given, therefore (*a*) BH is alſo given. And ſeeing that as CG is to AB, ſo is GD to BH, (*b*) ſo alſo is the whole CD to the whole AH, the ratio of the ſaid CD to AH ſhall be alſo given.

a 2. *prop.*
b 12. 5.

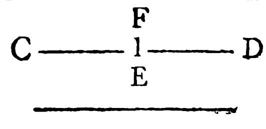
Again, ſeeing that the ſame CD is greater than EF by a given Magnitude, and in ratio; let the Magnitude DI be cut off therefrom: Therefore the ratio of the remainder CI to EF is given: Let the ſame be made of DI to FK. Therefore the ratio of the ſaid DI to FK ſhall be alſo given. But DI is given, therefore FK is alſo given. And ſeeing that as CI is to EF, ſo is ID to FK; ſo alſo is the whole (*c*) CD to the whole EK; the ratio c 12. 5.
of the ſaid CD to FK ſhall be given. But the ratio of the ſame CD to AH is alſo given: Therefore (*d*) the d 8. *prop.*
ratio of the ſaid AH to EK ſhall be given. And ſeeing that from the ſaid AH and EK, the given Magnitudes BH and FK are cut off, the Magnitudes AB and EF (*e*) e 15. *prop.*
are either in a given ratio to one another, or the one is greater than the other by a given Magnitude, and in ratio.

## PROP. XIX.

*If there are three Magnitudes* AB, CD, *and* E, *and that the firſt* AB *be greater than the ſecond* CD, *by a given Magnitude, and in ratio; and that the ſecond* CD *be greater than the third* E, *by a given Magnitude, and in ratio; alſo the firſt Magnitude* AB *ſhall be greater than the third* E, *by a given Magnitude, and in ratio*

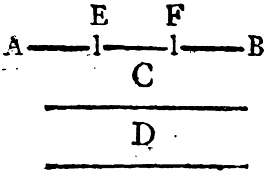


*Demonſtr.* For ſeeing that CD is greater than E by a given Magnitude, and in ratio; let the given Magnitude, CF be taken therefrom: Therefore the ratio of the remainder FD to E is given. Again, ſeeing that AB is greater than the ſame CD by a given Magnitude, and in ratio: Let the Magnitude AG be taken therefrom: Therefore the ratio of the remainder GB to CD is given: Let the ſame be made of GH to CF. Therefore the ratio of the ſaid GH to CF is given. But CF is given: Therefore alſo GH is given, and then AG is alſo given, the whole (*a*) AH ſhall a 3. *prop.*
be alſo given. But as GB is to CD, ſo is GH to CF, and ſo alſo (*b*) the remainder HB to b 19. 5.
the remainder FD: Therefore the ratio of the ſaid HB to FD is given. But the ratio of the ſame FD to E is alſo given: Therefore the ratio of HB to E is in [illegible]



c 11. *def.* given, and ſo is alſo the Magnitude AE: Wherefore the Magnitude AB (*c*) is greater than E by a given Magnitude, and in ratio.

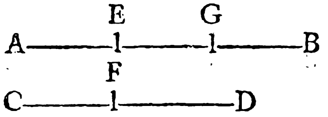
## OTHERWISE.



*Conſtruction.* Let there be three Magnitudes AB, C, and D, and let AB be greater than C by a given Magnitude, and in ratio; but let C be alſo greater than D, by a given Magnitude, and in ratio: I ſay, that AB is greater than D by a given Magnitude, and in ratio.

*Demonſtr.* Foraſmuch as AB is greater than C by a given Magnitude, and in ratio, let the given Magnitude AE be cut off therefrom: Therefore the ratio of the remainder EB to C is given. But the Magnitude C is greater than the Magnitude D by a given Magnitude, and in ratio; therefore (*d*) EB is greater than D by a d 13. *prop.* given Magnitude, and in ratio: Wherefore let the given Magnitude EF be cut off therefrom; and the ratio of the remainder FB to D ſhall be given. But AF is e 2. *prop.* (*e*) given. Therefore AB is greater than D by a given Magnitude, and in ratio.

## PROP. XX.

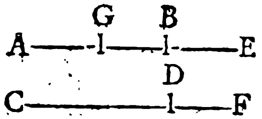


*If there are two given Magnitudes,* AB *and* CD, *and that from them there are taken Magnitudes* AE *and* CF, *having to one another a given ratio; either the remaining Magnitudes* EB *and* FD, *ſhall have to one another given ratio's; or elſe the one ſhall be greater than the other by a given Magnitude, and in ratio.*

*Demonſtr.* For ſeeing that both the Magnitudes AB and CD, are given, the ratio of the ſaid AB to CD is (*a*) alſo a 1. *prop.* given; and if it be the ſame as of AE to CF, that of the remainder EB to the remainder FD ſhall be (*b*) alſo the b 19. 5. ſame; and therefore the ratio of the ſaid EB to FD ſhall be alſo given. But if it be not the ſame, let it be ſo as that AE be to CF, as AG to CD. Now the ratio of the ſaid AE to CF is given: Therefore the ratio of the ſaid

AG to CD is given. But CD is given, therefore (*c*) AG c 3 *prop.*
is alſo given. But the whole AB is likewiſe given,
therefore (*d*) the remainder BG is given. And ſeeing that d 4. *prop.*
as AE is to CF, ſo is AG to CD, and alſo the remainder
EG to the remainder FD, the ratio of the ſaid EG to
FD is given. But GB is alſo given: Therefore the Magnitude
EB is greater (*e*) than the Magnitude BD by a given e 11. *def.*
Magnitude, and in ratio.

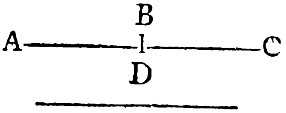
## PROP. XXI.



*If there are two Magnitudes given,* AB *and* CD, *and to them are added other Magnitudes* BE *and* DF, *having to one another a given ratio; either the whole* AE *and* CF *ſhall have to one another a given ratio, or elſe the one ſhall be greater than the other by a given Magnitude, and in ratio.*

*Demonſtr.* For ſeeing that both the Magnitudes AB
and CD are given: their ratio (*a*) is alſo given, and if a 1. *prop.*
it be the ſame ratio as of BE to DF, the ratio of the
whole AE to the whole CF ſhall be alſo given; for it
ſhall be (*b*) the ſame. But if it be not the ſame, let it b 12. 5.
be as BE is to DF, ſo BG to CD: Therefore the ratio
of the ſaid BG to CD is given. But CD is given;
therefore (*c*) alſo BG ſhall be given. But the whole AB
is given; therefore alſo the (*d*) remainder AG ſhall be given. c 2. *prop.*
And ſeeing that as BE is to DF, ſo is BG to CD, d 4. *prop.*
and alſo (*e*) the whole GE to the whole CF, the ratio of
the ſaid GE to CF ſhall be likewiſe given. But AG is given; e 12. 5.
therefore the Magnitude AE is grearer than the
Magnitude CF by a given Magnitude, and in ratio.

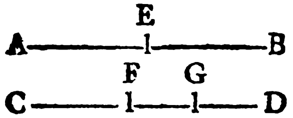
## PROP. XXII.



*If two Magnitudes* AB *and* BC, *have to ſome other Magnitude* D, *a given ratio, alſo their compound Magnitude* AC, *ſhall have to the ſame Magnitude* D, *a given ratio.*

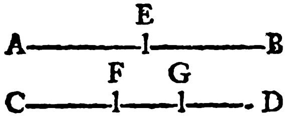
*Demonſtr.* For ſeeing that each Magnitude AB and
BC hath a given ratio to D, the ratio (*a*) of AB to
BC is given; and by compounding, (*b*) the ratio of AC a 8 *prop.*
to BC is given. But that of BC to D is alſo given, b 6. *prop.*
therefore (*c*) the ratio of the ſaid AC to D ſhall be like- c 8. *prop.*
wiſe given.

## PROP. XXIII.

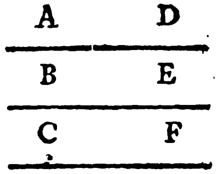


*If the whole* AB *be to the whole* CD *in a given ratio, and that the parts* AE *and* EB *be to the parts* CF *and* FD *in given ratio's, altho' they be not the ſame, the whole* (*to wit,* AB, AE, *and* BE,) *ſhall be to the whole* (*to wit,* CD, CF, *and* FD,) *in given ratio's.*

*Demonſtr.* For ſeeing that AE is to CF in a given ra-
tio, let the ſame be made of AB to CG ; therefore the
ratio of the ſaid AB to CG is given ; and conſequently
a 19. 5. alſo that (*a*) of the reſt EB to the reſt FG. But the ratio
of FD to the ſame EB is alſo given : Therefore the ratio
b 8. *prop.* of FD to FG (*b*) is likewiſe given, and therefore (*c*) that
c 5. *prop.* of FD to the remainder GD is alſo given. But the ratio
of AB to each of the Magnitudes CD and CG is given :
d 8. *prop.* Therefore (*d*) alſo the ratio of CD to CG is given, and
e 5. *prop.* again (*e*) that of CD to the remainder GD. But the ra-
f 8. *prop.* tio of FD to DG is given, therefore alſo (*f*) that of the
g 5. *prop.* ſame CD to FD, and conſequently that of (*g*) CD to the
remainder FC ; and therefore alſo the ratio of CF to FD ſhall be given. But the ratio of EB to FD is propoſed to be given ; therefore the ratio of CF to EB ſhall be given. Again, for that the ratio of AB to CD is given ; and alſo that of CD to each of thoſe FC and FD, the ratio of the ſame AB
h 8. *prop.* to each of the ſaid FC and (*h*) FD, ſhall be likewiſe given.
But the ratio of the ſaid FD to EB is given : Therefore
the ratio of AB to BE ſhall be alſo given, and conſe-
i 5. *prop.* quently AB to the remainder (*i*) AE. Wherefore by divi-
k *ſch.* 5. *pr.* ſion (*k*) the ratio of AE to EB ſhall be likewiſe given.
But the ratio of EB to FD is given. Therefore alſo
that of AE to FD. In like manner, ſeeing that the ra-
tio of CD to AB is given ; and that of AB to each of
his parts AE and EB ; alſo the ratio of the ſaid CD
l 8. *prop.* to each of the ſaid AE and EB, (*l*) ſhall be given : Where-
fore each of the Magnitudes AB, CD, AE, EB, CF, and
FD, is to each of the others in a given ratio.



## PROP. XXIV.



*If of three right-lines* A, B, *and* C, *proportional* A *to* B, *as* B *to* C, *the firſt* A *hath to the third* C *a given ratio, it will alſo have to the ſecond* B *a given ratio.*

*Demonſtr.* For, let there be expoſed another right-line D, and ſeeing that the ratio of A to C is given; let the ſame be made of D to F; therefore the ratio of D to F is given. But D is given, therefore F is alſo given; betwixt the two right-lines D and F, let there be taken (*a*) a mean proportional E. Therefore the rectangle made under D and F is equal (*b*) to the ſquare of E. But the ſame rectangle of D and F is (*c*) given: (for all the angles of that rectangle are given, being right-angles, and the ratio's that the ſides have to one another are alſo given;) therefore the ſquare of E is given, and conſequently the ſame right-line E is alſo given (for one equal thereto may be found, (*d*) ſeeing that the rectangle of D and F is given.) But D is given, therefore (*e*) the ratio of D to E is given, and as A is to C, ſo D is to F. But as A is to C, (*f*) ſo is the ſquare of A to the rectangle of A and C, and alſo as D is to F, ſo is the ſquare of D to the rectangle of D and F. Therefore as the ſquare of A is to the rectangle of A and C, ſo is the ſquare of D to the rectangle of D and F. But the rectangle of A and C is equal to the ſquare of B, (ſeeing that A, B, and C, are proportional) and that of D and F to the ſquare of E, therefore as the ſquare of A is to the ſquare of B, ſo is the ſquare of D to the ſquare of E: Wherefore (*g*) as A is to B, ſo is D to E. But the ratio of D to E is given, therefore (*h*) alſo the ratio of A to B is given.

a 13. 7.
b 17. 6.
c 3. *def.*
d 14. 2.
e 1. *prop.*
f 1. 6.
g 12. 6.
h 2. *def.*

## PROP. XXV. Plate VII. *Fig.* 29.

*If two lines* AB *and* CD, *given by poſition do interſect, the point* E *in which they cut one another, is given by poſition.*

*Demonſtr.* For if it change its place the one or the other of the lines AB and CD, would change its poſition: But ſo it is that by Suppoſition it changeth not: Therefore (*a*) the point E is given by poſition.

a 4. *def.*

PROP.

## PROP. XXVI.

A———B *If the extremeties* A *and* B, *of a right-line* AB, *are given in poſition, that ſame right-line* AB *is given in poſition and in magnitude.*

*Demonſtr.* For if the point A remaining in its place, the poſition, or the magnitude of the right-line AB ſhall change, the point B will fall elſewhere. But ſo it is, that by Suppoſition it doth not fall elſewhere. Therefore the right-line AB is given in poſition, and in magnitude.

## PROP. XXVII.

A———B *If one of the extremes* A *of a right-line* AB, *given in poſition and magnitude, be given, the other extremity* B *ſhall be alſo given.*

*Demonſtr.* For if the point A remaining in its place, the point B ſhall change and fall in ſome other place, either the poſition of the right-line AB, or its magnitude would change: But ſo it is that according to the Suppoſition, neither the one nor the other doth change. Therefore the point B is given.

## OTHERWISE.

*Conſtr.* On the center A, (*Fig.* 30.) with the diſtance AB, deſcribe the circumference BC.

a 6. *def.* *Demonſtr.* Therefore (*a*) that circumference BC is given by poſition. But the right-line AB is alſo
b 25. *prop.* given by poſition; therefore the point (*b*) B is given.

## PROP. XXVIII. Plate VII. *Fig.* 31.

*If through the given point* A, *there be drawn a right-line* DAE, *againſt another right-line* BC, *given in poſition, the right-line* DAE *ſo drawn, is given in poſition.*

*Demonſtr.* For if it be not given, the point A remaining in its place, the poſition of the right-line DAE may change: Let it then change if it be poſſible, and fall elſewhere, remaining parallel to BC, and let it be the line FAG: Therefore BC is parallel to the ſaid line FAG,

FAG. But (*a*) the ſame BC is alſo parallel to DAE: a 13. *def.*
Therefore (*b*) DAE is parallel to the ſaid line FAG, b 30. 1.
which is abſurd; ſeeing they join together, and meet in A: Therefore the poſition of the right-line DAE falls not elſewhere. Wherefore the ſaid line DAE is given in poſition.

## PROP. XXIX. Plate VII. *Fig.* 32.

*If to a right-line* AB, *given in poſition, and to a point* C *given therein, there be drawn a right-line* CD, *which ſhall make a given angle* ACD, *the line drawn* CD *is given in poſition.*

*Demonſtr.* For if it be not given in poſition, the point C remaining in its place, the poſition of the line CD obſerving the Magnitude of the angle ACD, will fall elſewhere. Let it fall elſewhere then if it be poſſible, and let it be CE. Therefore the angle ACD is equal to the angle ACE, the greater to the leſſer, which is abſurd. Therefore the poſition of the right-line CD, ſhall not fall elſewhere; and therefore the ſaid line CD is given in poſition.

## PROP. XXX. *Fig.* 33.

*If from a given point* A, *be drawn to a right-line* BC, *given in poſition, a right-line* AD, *making a given angle* ADB, *that line drawn* AD *is given in poſition.*

*Demonſtr.* For if it be not given, the point A remaining in its place, the poſition of the right-line AD changing, the magnitude of the angle ADB, will change. Let it change then, and let it be the right-line AE: Therefore the angle ADB is equal to the angle AEB, the greater (*a*) to the leſſer, which is abſurd. Therefore a 16. 1.
the poſition of the right-line AD doth not change; and therefore the ſaid line AD is given in poſition.

## OTHERWISE.

*Conſtr.* Through the point A let there be drawn the line EAF, parallel to the right-line BC.

*Demonſtr.* Then ſeeing that through the given point A, and againſt the right-line BC, given in poſition, there is

drawn the right-line EF, (*Fig.* 34.) thoſe lines EF and BC are parallels. But on the ſame lines doth alſo fall
b 29. 1. the right-line AD. Therefore (*b*) the angle FAD is equal to the given angle ADB; and therefore it is alſo given. Wherefore to the right-line EF given in poſition, and to the given point A therein, there is drawn the right-line
c 29. *prop.* AD, making the given angle FAD. Therefore (*c*) the ſaid line AD is given in poſition.

## OTHERWISE.

*Conſtr.* In the line BCE, (*Fig.* 35.) let there be taken the given point C, and through the ſame let there be drawn the line CF, parallel to the ſaid DA.

*Demonſtr.* Foraſmuch as AD and CF are parallels, and that on them there falls the right-line BCE, the
d 29. 1. angle FCB is equal (*d*) to the given angle ADB; and therefore it is alſo given. And ſeeing that the right-line BC is given in poſition, and that to a given point C therein, there is drawn the right-line FC, making the
e 29. *prop.* given angle FCB, that ſame line FC (*e*) is given in poſition. But through the given point A, oppoſite to the line FC given in poſition, there is drawn the line
f 28. *prop.* AD. Therefore the ſaid line (*f*) AD is given in poſition.

## OTHERWISE.

*Conſtr.* In the right-line BC (*Fig.* 36.) aſſume ſome point at F, and draw AF.

*Demonſtr.* Foraſmuch as each point A and F is given,
g 26. *prop.* the right-line AF is given (*g*) in poſition. But the line BC is alſo given in poſition. Therefore * the angle AFD is given. But by ſuppoſition, the angle ADF
h 32. 1. is given: Therefore DAF (which is the reſidue (*h*) of two right-angles) is given; and ſeeing that to the right-line AF given in poſition, and to the given point therein A there is drawn the right-line DA, making the given
i 29. *prop.* angle DAF, (*i*) that ſame line DA is given in poſition.

### Scholium.

* *EUCLIDE ſuppoſeth here that two right-lines being given in poſition, and inclining to one another, make a given angle, which ſome demonſtrate after this manner.*

*Demonſtr.* Foraſmuch as the two right-lines given in poſition, incline to one another, the inclination of thoſe lines is given. But the angle is the inclination of the lines: Therefore the angle which makes the right-lines given in poſition, and inclining to one another, is given.

*Another thus demonſtrates it.*

*Conſtr.* Let there be two right-lines inclining to one another, as AB and CB, (*Fig.* 37, 38.) given in poſition, and in the line AB let there be taken a given point A, and in BC alſo ſome point, as C; and let the right-line AC be drawn.

*Demonſtr.* Seeing that as well the point B, as each of
the points A and C is given, (*k*) the three right-lines k 26. *prop.*
AB, BC, and AC, are given in Magnitude. Wherefore of three direct lines equal unto them, a triangle may be conſtituted: Let there then be made the triangle FDE, having the ſide FD equal to the ſide AB, the ſide FE equal to the ſide AC, and the baſe DE equal to the baſe BC.

Seeing then the angles compriſed of equal right-lines
are equal, we have found the angle FDE equal to the
angle ABC; and therefore the ſame (*l*) angle ABC is l 1. *def.*
given.

## PROP. XXXI. Plate VII. *Fig.* 39.

*If from a given point* A *there be drawn to a right-line given in poſition* BC, *a right-line* AD, *given in magnitude, that line* AD *ſhall be alſo given in poſition.*

*Conſtr.* From the center A, with the diſtance AD, let the circle DEF be deſcribed.

*Demonſtr.* Foraſmuch as the center A is given in poſi-
tion, and the ſemidiameter AD in magnitude, the circle
DEF (*a*) is given in poſition. But the right-line BC is a 6. *def.*
alſo given in poſition: Therefore the point of interſection
D (*b*) is given, and ſeeing that the point A is alſo given; b 25. *prop.*
(*c*) the right-line AD is given in poſition. c 26. *prop.*

## PROP. XXXII. *Fig.* 40.

*If unto parallel right-lines* AB *and* CD, *given in poſition, there be drawn a right-line* EF, *making the given angles* BEF *and* EFD, *the line drawn* EF *ſhall be given in magnitude.*

*Conſtr.* For let there be taken in the line CD a given point G, and from that point let be drawn GH parallel to FE.

*Demonſtr.* Foraſmuch as the lines EF and HG are
a 29. 1. parallels, and that on them falls the line CD; (*a*) the
angle EFD is equal to the angle FGH. But the angle
EFD is given, therefore the angle FGH is alſo given.
And foraſmuch as to the right-line CD given in poſition,
and to the point G given in the ſame, there is drawn the
b 29. *prop.* right-line GH, making the given angle FGH, (*b*) the ſaid
line GH is given in poſition. But AB is alſo given in
c 25. *prop.* poſition, therefore (*c*) the point H is given. But the point
d 26. *prop.* G is alſo given: Therefore (*d*) the line GH is given in
e 34. 1. Magnitude, and is (*e*) equal to EF. Wherefore (*f*) the
f 1. *def.* ſaid line is given in Magnitude.

## P R O P. XXXIII. Plate VII. *Fig* 41.

*If unto parallel right-lines* AB *and* CD, *given in poſition, there be drawn a right-line* EF *given in magnitude, that line* EF *ſhall make the given angles* BEF *and* DFE.

*Conſtr.* For let there be taken in the right-line AB the point G, and through that point let there be drawn the line GH parallel to EF.

*Demonſtr.* Therefore EF is equal to the ſaid (*a*) GH.
a 34. 1. But EF is given in Magnitude, therefore GH is alſo
given in Magnitude. But the point G is given, and
therefore if on that point, with the diſtance GH, there
b 6. *def.* be deſcribed a circle, (*b*) that circle ſhall be given in po-
ſition: Let it be then deſcribed, and let it be HKL, the
ſaid circle HKL is therefore given in poſition. But the
line CD which cuts the circumference KHI, in H,
is alſo given in poſition. Therefore the ſaid point of
c 25. *prop.* interſection H (*c*) is given. But the point G is given:
d 26. *prop.* Therefore (*d*) the right-line GH is given in poſition. But
e *ſch.* 30. the right-line CD is alſo given in poſition: Therefore
*prop.* (*e*) the angle GHF is given. But to that angle (*f*) the
f 29. 1. angle EFD is equal; Therefore the angle EFD is given;
and therefore alſo the angle BEF; for that is only the
g 29. 1. reſidue of the ſum of two (*g*) right-angles.

## O T H E R W I S E.

*Conſtr.* Let there be taken in the right-line CD, (*Fig.* 42.) the point G, and let GD be put equal to EF, then from the center G, with the diſtance GD, let there be deſcribed the circle HDB, and draw GB.

*Demonſtr.* Foraſmuch as the center G is given in poſition, and the ſemidiameter GD in magnitude, the circle BDH (*h*) is given in poſition. But the line AB is alſo given in poſition: Therefore (*i*) the point B is given. But the point G is alſo given, therefore (*k*) the right-line GB is given in poſition. But the right-line CD is alſo given in poſition: Therefore (*l*) the angle BGD is given. Wherefore if EF be parallel to BG, the angle EFD (*m*) ſhall be given, and conſequently alſo the other angle BEF. But the right-lines BG and EF being not parallels, let them meet in the point H. Foraſmuch as EB is parallel to FG, and EF is equal to GD, that is to ſay, to BG; alſo FH (*n*) ſhall be equal to GH (for EH and BH being cut proportionally (*o*) by the parallel FG, as EF is to FH, ſo is BG to GH, and by permutation, as EF is to BG, ſo is FH to GH:) Therefore (*p*) the angle HFG is equal to the angle HGF, but the ſaid angle HGF is given (for it is equal (*q*) to the given angle BGD:) Therefore the angle HFG is alſo given. But to that angle the angle BEF is equal; and therefore is given, as alſo the remaining angle EFG.

h 6. *def.*
i 25. *prop.*
k 26. *prop.*
l *ſch.* 30. *prop.*
m 29. 1.
n 14. 5.
o 2. 6.
p 5. 1.
q 15. 1.

## PROP. XXXIV. Plate VII. *Fig.* 43.

*If from a given point* E, *there be drawn unto parallel right-lines* AB *and* CD, *given in poſition, a right-line* EFG, *that right-line* EFG *ſhall be divided in a given ratio* (*to wit*) *as* EF *to* FG.

*Conſtr.* For from the point E let there be drawn the line EH, perpendicular to the line CD.

*Demonſtr.* Foraſmuch as from the given point E there is drawn to the line CD the right-line EH, making the given angle EHG, (*a*) the ſaid line EH is given in poſition, but both the one and the other lines AB and CD is alſo given in poſition. Therefore (*b*) the points of interſection K and H, are given. But the point E is alſo given: Therefore (*c*) each line EK and KH is given. Wherefore (*d*) the ratio of the ſaid EK to KH is given. But as EK is to KH, ſo is EF to FG; (for in the triangle GEH the line KF being parallel to HG, the ſides EH and EG are cut proportionally:) Therefore the ratio of the ſaid EF to FG is given.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d 1. *prop.*

## OTHERWISE. Plate VII. *Fig.* 44.

*Conſtr.* To the parallel right-lines given in poſition, AB and CD, let there be drawn from the point E the right-line FEG : I ſay, that the ratio of GE to EF is given.

*Demonſtr.* For from the point E let there be drawn
to CD the perpendicular EH, and produced to the point
K ; ſeeing therefore that from the point E to the right-
line CD, given in poſition, there is drawn the line EH,
a 30. *prop.* making the given angle EHG, (*a*) the ſaid line EH is
given in poſition. But each line AB and CD is alſo
b 25. *prop.* given in poſition : Therefore (*b*) each point of interſection
H and K is given. But the point E is alſo given, there-
c 26. *prop.* fore (*c*) each of the lines EH and IK is given in Magni-
d 1. *prop.* tude ; and therefore (*d*) the ratio of the ſaid EH to EK
e 4. 6. is given. But (*e*) as EH is to FK ſo is EG to EF (for
the oppoſite angles at the point E being equal, and
the lines AB and CD parallels, the triangles EHG and
EKF are equiangled ; and therefore as EH is to EG,
ſo is EK to EF ; and by permutation as EH to EK, ſo
is FG to EF.) Therefore the ratio of the ſaid lines EG
to EF is given.

## PROP. XXXV. *Fig.* 45.

*If from a given point* A, *to a right-line* BC, *given in poſition, there be drawn a right-line* AD, *which let be divided in* E, *in a given ratio* (*to wit*) *as* AE *to* ED, *and that by the point of ſection* E *there be drawn a right-line* FEG, *oppoſite to the right-line* BC, *given in poſition, the line* FG *ſo drawn ſhall be given in poſition.*

*Conſtr.* For from the point A, let there be drawn the line AH, perpendicular to the line BC.

*Demonſtr.* For ſeeing that from the given point A
there is drawn to BC given in poſition, the right-line
a 30. *prop.* AH making the given angle AHD, (*a*) the ſaid line AH
is given in poſition. But BC is alſo given in poſition :
b 25. *prop.* Therefore (*b*) the point H is given. But the point A is
c 26. *prop.* alſo given : Therefore (*c*) the line AH is given in mag-
d 2. 6. nitude and in poſition. And ſeeing that (*d*) as AE is
to ED, ſo is AK to KH, and that the ratio of AE to
ED is given, alſo the ratio of AK to KH is given ;
e 6. *prop.* and by compounding, (*e*) the ratio of AH to AK is given.
f 2. *prop.* But AH is given in Magnitude : Therefore (*f*) alſo

AK is given in Magnitude. But AK is alſo given in poſition, and the point A is given: Therefore (*g*) the point K is alſo given, and ſeeing that by the ſaid given point K there is drawn the line FG, oppoſite to the right-line BC given in poſition; the ſaid line FG (*h*) is given in poſition.

g 27. *prop.*
h 28. *prop.*

## PROP. XXXVI. Plate VII. *Fig.* 46.

*If from a given point* A, *there be drawn to a right-line* BC *given in poſition, a right-line* AD, *and to it be added a right-line* AE, *having to the ſame* AD *a given ratio, and that through the extremity* E *of the added line* AB, *there be drawn a right-line* FEK, *oppoſite to the line* BC, *given in poſition, that ſame line* FEK *ſhall be given in poſition.*

*Conſtr.* For from the point A let there be drawn to the line BC, the perpendicular AL, and let it be prolonged to the point G.

*Demonſtr.* Foraſmuch as from the given point A, there is drawn to the right-line BC, given in poſition, the right-line GL, which makes the given angle GLD, (*a*) that line GL is given in poſition. But BC is alſo given in poſition, therefore (*b*) the point L is given; and ſeeing that the point A is alſo given, the line (*c*) AL is given. But foraſmuch as the ratio of AE to AD, is given, and that (*d*) as the ſaid AE is to AD, ſo is AG to AL; (becauſe the triangles ALD and AGE are equiangled) the ratio of AG to AL is alſo given. But AL is given in Magnitude: Therefore (*e*) AG is given in Magnitude. But it is alſo given in poſition, and the point A is given: Therefore (*f*) the point G is alſo given. And ſeeing that by the ſame given point G there is drawn the line FK, oppoſite to the right-line BC, given in poſition, (*g*) the ſaid line FK is given in poſition.

a 30. *prop.*
b 26. *prop.*
c 26. *prop.*
d 4. 6.
e 2. *prop.*
f 27. *prop.*
g 28. *prop.*

## PROP. XXXVII. *Fig.* 47.

*If unto parallel right-lines* AB *and* CD, *given in poſition, there be drawn a right-line* EF, *divided in the point* G, *in a given ratio,* (*to wit,*) *of* EG *to* GF: *and if through the point of ſection* G, *there be drawn oppoſite to the right-lines* AB *or* CD, *given in poſition, a right-line* H G K, *that line ſhall be given in poſition.*

*Conſtr.* For let there be taken in the line AB the given point L, and from that point let there be drawn the line LN, perpendicular to CD.

*Demonſtr.* Seeing that from the given point L, there is drawn to the right-line CD, the line LN, making the a 30. *prop.* given angle LND, the ſaid LN (*a*) is given in poſition. But CD is alſo given in poſition: Therefore the point b 25. *prop.* N (*b*) is given. But the point L is alſo given: Therefore c 26. *prop.* (*c*) the line LN is given; and ſeeing that the ratio of FG to GE is given, and that * as FG is to GE, ſo is NM to to ML, the ratio of the ſaid NM to ML is given; and d 6. *prop.* by compounding, (*d*) the ratio of LN to LM is alſo given. But LN is given in Magnitude; therefore ML e 2. *prop.* is (*e*) given in Magnitude. But it is alſo given in poſition, and the point L is given; Therefore the point M f 27. *prop.* (*f*) is alſo given. And conſidering that through the ſaid point M there is drawn the right-line KH, oppoſite to the right-line CD, given in poſition, the ſaid line KH is alſo given in poſition.

### Scholium.

* *EUCLIDE ſuppoſeth here, that as* FG *is to* GE, *ſo* NM *is to* ML; *but by another it is thus demonſtrated.*

*The lines* EF *and* LN *are parallels or not parallels: Let them in the firſt place be parallels, and foraſmuch as by Conſtruction the lines* EL, FN, EF, *and* LN, *are parallels,* EN *ſhall be a parallelogram, and therefore the ſide* EF *is equal to the ſide* LN. *Again, ſeeing that* MG *is parallel to* NF, *and* GF *to* MN, GN *ſhall be alſo a parallelogram; and therefore the ſide* GF *is equal to the ſide* MN. *Wherefore the equal ſides* EF *and* LN, *ſhall have to the equal ſides* FG g 7. 5. *and* MN, (g) *one and the ſame ratio. Therefore as* EF *is* h 17. 5. *to* FG, *ſo is* LN *to* MN; *and by dividing,* (h) *as* GE *to* GF, *ſo is* LM *to* MN.

*Now ſuppoſe that the lines* EF *and* LN (Fig. 48.) *are not parallels, but that they meet in the point* O. *Foraſmuch as in the triangle* OFN *there is drawn* HK, *pa-* i 2. 6. *rallel to* FN *one of the ſides;* (i) *the ſides* OF *and* ON *are divided proportionally; and therefore as* FG *is to* GO, *ſo is* NM *to* MO. *Again, ſeeing that in the triangle* OGM *there is drawn* EL, *parallel to the ſide* GM, *the ſides* OG *and* OM *are divided proportionally: Where-* k 2. 6. *fore* (k) *as* OE *is to* EG, *ſo is* OL *to* LM, *and by* l 18. 5. *compounding,* (l) *as* OG *is to* EG, *ſo is* OM *to* LM; *but it hath been demonſtrated that as* FG *is to* GO, *ſo* m 22. 5. *is* NM *to* MO; *therefore in ratio of equality,* (m) *as* FG *is to* GE, *ſo is* NM *to* ML.

PROP.

## PROP. XXXVIII. Plate VII. *Fig* 49.

*If unto parallel right-lines* AB *and* CD, *there be drawn a right-line* EF, *and that to it there be added ſome other right-line* EG, *which hath a given ratio to the ſame* EF; *and if through the extremity G of the added line* EG, *there be drawn a right-line* HK, *againſt the parallels given in poſition* AB *and* CD, *the line drawn* HK *ſhall be alſo given in poſition.*

*Conſtr.* For let there be taken in the line AB, the given point N, and from thence let there be drawn to CD the perpendicular NM, and let it be prolonged to the point L.

*Demonſtr.* Foraſmuch as from the given point N there
is drawn to the right-line CD, given in poſition, the
right-line NM making a given angle NMF, the ſaid
angle NMF (*a*) is given in poſition. But the line CD a 30. *prop.*
is alſo given in poſition: Therefore (*b*) the point M is b 25. *prop.*
given. But the point N is alſo given: Therefore (*c*) the c 26. *prop.*
line NM is given, and therefore the ratio of EG to EF
is given; and becauſe (*d*) as EG to EF, ſo is LN to MN, d *ſch.* 37.
the ratio of LN to NM is alſo given: But NM is given, *prop.*
therefore LN is (*e*) alſo given. But the point N is e 2. *prop.*
given: Therefore (*f*) the point L is alſo given. Seeing f 27. *prop.*
then that by the given point L there is drawn the right-
line HK, oppoſite to the line AB given in poſition, (*g*) g 28. *prop.*
the ſaid line AK is alſo given in poſition.

## PROP. XXXIX. *Fig.* 50.

*If all the ſides of a triangle* ABC *are given in magnitude, the triangle is given in kind.*

*Conſtr.* For, let there be expoſed the right-line DG given in poſition, ending in the point D; but being infinite towards the other part G, and therein let be taken DE, equal to AB.

*Demonſtr.* Now ſeeing the ſaid AB is given in magni-
tude, DE is ſo alſo; but the ſame DE is alſo given in
poſition, and the point D is given: Therefore (*a*) the a 27. *prop.*
point E is given.

Again, Let EF be put equal to BC; and ſeeing that
BC is given in magnitude, EF ſhall be ſo alſo. But the
ſaid EF is in like manner given in poſition, and the
point E is given: Therefore (*b*) the point F is given. b 27. *prop*

Furthermore, Let FG be taken equal to AC. Now foraſmuch as the ſaid AC is given in magnitude, FG is

ſo alſo. But FG is alſo given in poſition, and the point
F is given : Therefore the point G is alſo given. Now
from the center E, with the diſtance ED, let there be de-
c 6. *def.* ſcribed the circle DHK, (*c*) and that circle ſhall be given
in poſition. Again, on the center F, and diſtance FG,
d 6. *def.* let there be deſcribed the circle GLK. Therefore (*d*) the
e 25. *prop.* ſaid circle GLK is given in poſition ; and therefore (*e*) the
f 26. *prop.* point of Interſection K is given. But each of the points
E and F is given : Therefore each line (*f*) EK, EF, and
FK, is given in poſition and magnitude. Therefore the
triangle FK is given * in kind ; but it is equal and
alike to the triangle ABC ; and therefore the triangle
ABC is alſo given in kind.

### Scholium.

* *EUCLIDE ſuppoſeth here, that a triangle whoſe ſides
are given in magnitude and poſition, is given in kind ;
but the antient Interpreters demonſtrated it in a manner
thus. Foraſmuch as the right-lines* KE *and* EF *are gi-*
g 1. *prop.* *ven,* (g) *the ratio which they have to one another is alſo
given. Alſo the right-lines* EF *and* FK *being given,
their ratio is alſo given ; and in like manner, the ratio
of the ſaid* EK *and* FK *is given. Again, ſeeing that*
h *ſch.* 30. *prop.* *the ſame lines* KE *and* EF *are given in poſition,* (h)
*the angle* KEF *is given in magnitude : Moreover, the
right-lines* EF *and* FK *being given in poſition, the an-
gle* EFK *is given in magnitude, as is alſo the reſidue*
EKF, *and ſo in the triangle* EKF *are all the angles*
i 15. *def.* *given, and alſo the ratio's of the ſides : Therefore* (i) *the
ſaid triangles* EKF *is given in kind.*

### PROP. XL. Plate VII. *Fig.* 37, 38.

*If the angles of a triangle* ABC, *are given in magnitude, the triangle is given in kind.*

*Conſtr.* Let there be expoſed the right-line DE, given in poſition and in magnitude ; and let there be conſtitued at the point D the angle EDF, equal to the angle CBA, and at the point E the angle DEF, equal to the angle BCA ; therefore the third angle BAC is equal to the third angle DFE.

*Demonſtr.* For each of the angles conſtituted in the points A, B, and C, is given: Therefore each of thoſe which are poſited in the points D, F, and E, is alſo given; and ſeeing that to the right-line DE given in poſition, and to the point D given therein, there is drawn the right-line DF, which makes the given angle EDF, (*a*) the line DF is given in poſition; and for the ſame reaſon the line EF is given in poſition: Therefore (*b*) the point F is given in poſition. But each of the points D and E is given: Therefore (*c*) each of the lines DF, DE, and FF is given in magnitude. Wherefore the triangle DFE is given in kind; and is alike to the triangle ABC: Therefore the triangle ABC is given in kind.

a 29. *prop.*
b 25. *prop.*
c 26. *prop.*

## PROP. XLI. Plate VII. *Fig.* 37, 38.

*If a triangle* ABC, *hath one angle* BAC *given, and that the two ſides* BA *and* AC, *which conſtitute it, have to one another a given ratio, the triangle is given in kind.*

*Conſtr.* For, let there be expoſed the right-line DF given in magnitude and poſition. And thereon, and at the given point F, let there be conſtitued the angle DFE equal to the angle BAC.

*Demonſtr.* Now the angle BAC is given: Therefore alſo the angle DFE is given, and ſeeing that to the right-line DF given in poſition, and from the given point F therein is drawn a right-line FE, making the given angle DFE, (*a*) the ſaid line FE is given in poſition. But ſeeing that the ratio of AB to AC is given, let the ſame be made of DF to FE, then let DE be drawn. Therefore the ratio of DF to FE is given. But DF is given: Therefore (*b*) FE is given in magnitude. But the ſame FE is alſo given in poſition, and the point F is given. Therefore (*c*) the point E is alſo given. But each of the points D and F is given: Therefore (*d*) each of the right-lines DF, FE, and DE is given in poſition and magnitude. Wherefore (*e*) the triangle DEF is given in kind. And ſeeing that the two triangles ABC and DEF have an angle equal to an angle, that is to ſay, the angle BAC equal to the angle DFE, and the ſides which conſtitute thoſe equal angles, proportional; (*f*) the triangle ABC is alike to the triangle DEF. But the triangle DEF is given in kind: Therefore the triangle ABC is given in kind.

a 29. *prop.*
b 2. *prop.*
c 27. *prop.*
d 26. *prop.*
e 39. *prop.*
f 6. 6.

## PROP. XLII. Plate VIII. *Fig.* 1, 2.

*If the ſides of a triangle* ABC, *are to thoſe of another in given ratio's, the triangle* ABC *is given in kind.*

*Conſtr.* For, let there be expoſed the right-line D, given in magnitude, and ſeeing that the ratio of BC to AC is given, let the ſame be made of D to E.

a 2. *prop.* *Demonſtr.* Now D is given, therefore (*a*) E is alſo
given. Again, ſeeing that the ratio of AC to AB is
given, let the ſame be made of E to F. Now E is
b 2. *prop.* given, therefore (*b*) F is alſo given. Now of three right-
lines, equal to the three given right-lines D, E, and F,
(and of which three lines, two of them, in what manner
ſoever they be taken, are greater than the other,)
let there be conſtituted the triangle GHK, in ſuch ſort
as D may be equal to HK; but E is equal to KG, and
GH, equal to F; therefore each of the ſaid lines HK, KG
c 39. *prop.* and GH, is given in magnitude: Wherefore (*c*) the triangle
HGK is given in kind, And ſeeing that as BC is to
CA, ſo is D to E, and that D is equal to HK, and E
to KG, as BC is to CA, ſo HK is to KG. Again,
ſeeing that as CA is to AB, ſo is E to F, and that E
is equal to KG, and F to GH; as CA is to AB, ſo is
KG to GH. But it hath been demonſtrated, that as
BC is to CA, ſo is HK to KG: Therefore in ratio of
equality, as BC is to AB, ſo is HK to GH. There-
d 5. 6. fore (*d*) the triangle ABC is alſo given in kind.

## PROP. XLIII. *Fig.* 3.

*If the ſides* BC *and* BA, *about one of the acute angles of a rectangled triangle* ABC, *have to one another a given ratio, that triangle is given in kind.*

*Conſtr.* Let there be expoſed the right-line DE given
in magnitude and poſition, and on it let there be de-
a 6. *def.* ſcribed the ſemicircle DGE: Therefore (*a*) the ſemicircle
DGE is given in poſition.

*Demonſtr.* For the line DE being given, and divided
in two equal parts, the center of the ſaid circle is given
in poſition, and the ſemidiameter in magnitude. And
foraſmuch as the ratio of BC to BA is given, let the
ſame be made of DE to F: Therefore the ratio of DE
b 2. *prop.* to F is given. But DE is given, therefore F (*b*) is alſo
c 19. 1. given. Now BC is greater than (*c*) AB: Therefore ED
d 14. 5. is (*d*) alſo greater than F. Let DG be fitted equal to F,
and let EG be drawn; then on the center D, with the
diſtance

diſtance DG, let the circle GK be deſcribed. Now
that circle (*e*) is given in poſition, ſeeing that the center e 6. *def.*
D is given, and the ſemidiameter DG is alſo given in
magnitude. But the ſemicircle DGE is alſo given in
poſition: Therefore (*f*) the point of interſection G is f 25. *prop.*
given. But the points D and E are alſo given, therefore
(*g*) each of the right-lines DE, DG, and EG is g 26. *prop.*
given in poſition and magnitude. Wherefore (*h*) the h 39. *prop.*
triangle DGE is given in kind. And ſeeing that the
triangles ABC and DGE have an angle equal to an angle,
to wit, the right angle BAC to the right-angle (*i*) DGE, i 31. 3.
and the ſides about the angles CBA and EDG proportional.
But each of the others ACB and DEG are leſs than
a right-angle: Therefore thoſe triangles ABC and DEG
(*k*) are alike. But the triangle DGE is given in kind: k 7. 6.
Therefore the triangle ABC is alſo given in kind.

## PROP. XLIV. Plate VIII. *Fig.* 4.

*If a triangle* ABC, *hath one angle* B *given, and that the ſides* BA *and* AC, *about another angle* BAC, *have to one another a given ratio, the triangle* ABC *is given in kind.*

*Conſtr.* Now the given angle B is either acute or obtuſe, (for it was a right-angle in the foregoing propoſition.) Let it be in the firſt place acute, and from the point A let AD be drawn perpendicular to BC.

*Demonſtr.* Therefore the angle ADB is given: But the
angle B is alſo given; and therefore the third angle
BAD is given: Wherefore (*a*) the triangle ABD is given a 40. *prop.*
in kind; and therefore (*b*) the ratio of BA to AD is given. b 3. *def.*
But the ratio of the ſame BA to AC is alſo given:
Therefore (*c*) the ratio of AD to AC is given, and the c 8 *prop.*
angle ADC is a right-angle: Wherefore the triangle (*d*) d 43. *prop*
ACD is given in kind: Therefore (*e*) the angle C is given. e 3. *def.*
But the angle B is alſo given; and therefore the other
angle BAC is given: Therefore (*f*) the triangle ABC is f 40. *prop.*
given in kind.

*Conſtr.* Now let the angle ABC be obtuſe, and on the ſide CB prolonged, let there be drawn the perpendicular AD.

*Demonſtr.* Foraſmuch as the angle ABC (*Fig.* 5.) is
given, the angle ABD, which follows it, ſhall be given.
But the angle ADB is alſo given: Therefore the third
angle DAB is given. Wherefore (*g*) the triangle ABD g 40. *prop.*
is given in kind; and therefore (*h*) the ratio of DA to h 3. *def.*
AB is given. But the ratio of AB to AC is alſo
given:

i 8. *prop.* given : Therefore (*i*) the ratio of DA to AC is given, and the angle D is a right-angle. Therefore the triangle DAC is given in kind, and therefore the angle ACB is given. But the angle ABC is alſo given : Therefore the third angle BAC is given. Wherefore the triangle ABC is given in kind.

## PROP. XLV. Plate VIII. *Fig.* 4.

*If a triangle* ABC *hath one angle* BAC *given, and that the line compounded of the two ſides* AB *and* AC, *about the ſaid given angle* BAC, *hath to the other ſide* BC *a given ratio, the triangle* ABC *is given in kind.*

*Conſtr.* For, let the angle BAC be divided into two
a 7. *prop.* equal parts by the line AD, therefore (*a*) the angle CAD is given.

b 3. 6. *Demonſtr.* Seeing that as AB is to AC, ſo (*b*) is BD to
c 18. 5. CD ; by compounding, (*c*) as the line compounded of CAB is to CA, ſo is BC to CD, and by permutation, as the line compounded of CAB is to CB, ſo is CA to CD. But the ratio of the line compounded of CAB to BC, is given ; therefore the ratio of CA to CD is alſo
d 44. *prop.* given, and the angle CAD is given. Therefore (*d*) the triangle ACD is given in kind, and therefore the angle C is given. But the angle BAC is alſo given : There-
e 40. *prop.* fore the third angle B is given : Wherefore (*e*) the triangle ABC is given in kind.

## OTHERWISE. *Fig.* 6.

*Conſtr.* Let BA be prolonged directly unto the point D, ſo that AD may be equal to AC, and let CD be joined.

*Demonſtr.* Foraſmuch as the ratio of the line compounded of CAB to CB is given, and that AD is equal to AC, the ratio of the whole line BD to BC is given. But the angle ADC is alſo given, for it is the half of the
f 32. 1. given angle BAC (for the ſaid angle BAC (*f*) is equal to
g 5. 1. the two internal angles ACD and ADC, which are (*g*) equal to one another, becauſe the ſides AC and AD are
h 44. *prop.* equal :) Wherefore the triangle BDC (*h*) is given in kind, and therefore the angle B is given. But the angle BAC is alſo given. Therefore the remaining angle ACB is gi-
i 40. *prop.* ven : Wherefore (*i*) the triangle ABC is given in kind.

## RPOP. XLVI. Plate VIII. *Fig.* 4.

*If a triangle* ABC *hath one angle* B *given, and that the line* CAB *compounded of the two ſides* AC *and* AB *about another angle* BAC *hath to the other ſide* BC *a given ratio, the triangle* ABC *is given in kind.*

*Conſtr.* For let the angle BAC be divided into two eqaal parts, by the line AD.

*Demonſtr.* Therefore (as hath been ſhewn in the foregoing propoſition) the compound line CAB is to CB, as AB is to BD. But the ratio of the ſaid compound line CAB to CB is given: Therefore alſo the ratio of AB to BD is given. But the angle B is alſo given: Therefore the triangle ABD (*a*) is given in kind; and a 41. *prop.* therefore (*b*) the angle BAD is given. But the angle b 2 *def.* BAC is double to that of BAD; and therefore it is alſo given. Therefore the third angle C is given. Wherefore the triangle ABC is given in kind.

## OTHERWISE. *Fig.* 6.

*Conſtr.* Let BA be prolonged directiy, and let AD be put equal to AC, and let CD be joined.

*Demonſtr.* Foraſmuch as the ratio of the line compounded of CAB to CB is given, and that AD is equal to AC, the ratio of BD to BC is given; and the angle B is alſo given: Therefore the triangle CDB (*c*) is given in kind; and therefore (*d*) the angle c 41. *prop.* D is given: Therefore the angle BAC which is double d 3 *def.* to BDC, is alſo given: Wherefore the other angle ACB is given; and therefore the triangle ABC is given in kind.

## PROP. XLVII. *Fig.* 7.

*Rectiline figures,* AB, CDE, *given in kind, are divided into triangles given in kind.*

*Conſtr.* For let the right-lines EB and EC be drawn.

*Demonſtr.* Foraſmuch as the rectiline figure ABCDE
a 3. *def.* is given in kind, the angle (*a*) BAE is given, and the
ratio of the ſide AB to AE is alſo given: Therefore
b 41. *prop.* (*b*) the triangle BAE is given in kind. Wherefore
the angle ABE is given. But the whole angle ABC
c 4. *prop.* is alſo given: Therefore (*c*) the remaining angle EBC is
given. But the ratio of the ſide AB to the ſide
BE, and alſo that of AB to BC is given: Therefore
d 8. *prop.* (*d*) the ratio of BC to BE is given, and the
angle CBE is alſo given: Therefore (*e*) the triangle
e 41. *prop.* BCE is given in kind. By the ſame way it may be
demonſtrated that the triangle CDE is given in kind.
Therefore rectiline figures given in kind divide themſelves into triangles given in kind.

## PROP. XLVIII. Plate VIII. *Fig.* 8.

*If on one and the ſame right-line* AB, *are deſcribed triangles, as* ACB *and* ABD, *given in kind, thoſe triangles ſhall have to one another a given ratio, as* ACB *to* ABD.

*Conſtr.* For from the points A and B, let there be drawn at right-angles on the line AB, the lines AE and BG, and prolonged unto the points F and H; through the points C and D, let there be drawn the lines ECG and FDH, parallel to AB.

*Demonſtr.* Foraſmuch then as the triangle ABC is
a 3. *def.* given in kind, (*a*) the ratio of CA to BA is given,
and the angle CAB is alſo given; but the angle BAE
is given: Therefore the remaining angle CAE is alſo
given; but the angle CAE is given; and therefore
b 40. *prop,* the other angle ACE is alſo given. Wherefore (*b*)
the triangle AEC is given in kind. Now the ratio
c 80. *prop.* of EA to AB (*c*) is given; (for (*d*) the ratio of EA
d 3. *def.* to AC, and that of AC to AB is given;) and in like
e 8. *prop.* manner, the ratio of FA to AB is given. Therefore
f 1. 6. (*e*) the ratio of EA to AF is given; but as AE is
g 41. 1. to AF, ſo is (*f*) the parallelogram AH to the parallelogram
h 15. 5. AG; but ACB is (*g*) the half of AH, and
ADB the half of AG; therefore the ratio of the
triangle ACB to the triangle of ADB is given; for
it is the ſame ratio with that of AH to AG (*h*);
that is to ſay, of EA to AF, which is given.

## PROP. XLIX. Plate VIII. *Fig.* 9.

*If on one and the ſame right-line* AB, *there are deſcribed any two rectiline figures* AECFD *and* ADB, *given in kind, they ſhall have to one another a given ratio (to wit)* AECFB *to* ADB.

*Conſtr.* For let the lines FA and FE be drawn :
Therefore each of the triangles (*a*) ABF, AFE, and a 47. *prop.*
ECF is given in kind.

*Demonſtr.* Seeing that on one and the ſame right-
line EF there are deſcribed the triangles ECF and EAF,
given in kind; the ratio of ECF to EAF (*b*) is given. b 48. *prop.*
Therefore by compounding, (*c*) the ratio of AECF to c 6. *prop.*
EAF is given. But the ratio of the ſaid EAF to FAB is
given, (*d*) becauſe they are triangles given in kind, deſcribed d 48. *prop.*
on one and the ſame right-line AF : There-
fore (*e*) the ratio of AECF to FAB is given. Where- e 8. *prop.*
fore by compounding, (*f*) the ratio of AECFB to f 6. *prop.*
FAB is given. But the ratio of the ſame FAB to g 48. *prop.*
ABD (*g*) is given : Therefore (*h*) the ratio of AECFD h 8. *prop.*
to ABD is alſo given.

## PROP. L. *Fig.* 10.

*If two right-lines* AB *and* CD, *have to one another a given ratio, and that on thoſe lines there be deſcribed rectiline figures* AEB *and* CFD, *alike, and alike poſited, they will have to one another a given ratio.*

*Demonſtr.* To the two lines AB and CD, let there
be taken a third proportional G. Therefore as AB is to
CD, ſo is CD to G. But the ratio of AB to CD is gi-
ven : Therefore the ratio of CD to G is alſo given :
Wherefore (*a*) the ratio of AB to G is given. But (*b*) as a 8. *prop.*
AB is to G, ſo is AEB to CFD : Therefore the ratio of b *cor.* 19.
the ſame AEB to CFD is given. 20. 6.

## PROP. LI. *Fig.* 11, 12.

*If two right-lines* AB *and* CD *have to one another a given ratio, and that upon them there be deſcribed any rectiline figures* AEB *and* CFD, *given in kind, they will have to one another a given ratio, (to wit, that of* AEB *to* CFD.)

*Conſtr.* For on AB let the rectangled figure AH be deſcribed alike, and alike poſited to DF.

*Demonſtr.* Now DF is given in kind: Therefore alſo AH is given in kind. But AEB is alſo given in kind, and deſcribed on the ſame line AB: Therefore (*a*) the ratio of AEB to AH is given: And ſeeing that the ratio of AB to CD is given, and that on thoſe lines are deſcribed the rectiline figures AH and DF alike, and alike poſited, the ratio (*b*) of the ſaid line AH to DF is given. But the ratio of AEB to AH is alſo given: Therefore the ratio (*c*) of AEB to DF is given.

a 49. *prop.*
b 50. *prop.*
c 8. *prop.*

## PROP. LII. Plate VIII. *Fig.* 7

*If on a right-line* BE, *given in magnitude, there be deſcribed a figure* BAE, *given in kind, that figure* BAE *is given in magnitude.*

*Conſtr.* For on the ſame line BE let the ſquare BD be deſcribed. Therefore BD is given in kind * and in magnitude.

*Demonſtr.* Seeing that on the right-line BE, are deſcribed the two rectiline figures DAE and BD, given in kind, (*a*) the ratio of BAE to BD is given: Therefore (*b*) BAE is given in magnitude.

a 29. *prop.*
b 2. *prop.*

## Scholium.

* *The antient interpreter hath noted here that every ſquare is given in kind; for that all the angles thereof are given; being all equal and right-angles: But alſo the ratio's of the ſides are given; for thoſe ſides being all equal, their ratio's are alſo equal. Moreover, whenſoever a ſquare is expoſed, a ſquare equal thereto may be exhibited; and therefore the ſquare is given in magnitude, as alſo each ſide thereof.*

## PROP. LIII. *Fig.* 13, 14.

*If there are two figures* AD *and* EH, *given in kind, and that one ſide* BD *of the one, hath to a ſide* FH *of the other, a given ratio; the other ſides ſhall have alſo to the other ſides given ratio's.*

*Demonſtr.* For ſeeing that the ratio of BD to FH is gi-
ven, and alſo that (*a*) of BD to BA, (*b*) the ratio of the a 3. *def.*
ſaid AB to FH is given. But the ratio of the ſame FH b 8. *prop.*
to EF (*c*) is alſo given: Therefore (*d*) the ratio of AB c 3. *def.*
to EF is given. In like manner alſo the ratio's of the d 8. *prop.*
other ſides to the other ſides are given

## PROP. LIV. Plate VIII. *Fig.* 15, 16.

*If two figures* A *and* B *given in kind, have to one another a given ratio, alſo their ſides ſhall be to one another in a given ratio*

*Conſtr.* For either the figure A is alike and alike poſited to B, or is not: Let it in the firſt place be alike, and alike poſited; and let there be taken the line G, a third proportional to the lines CD and EF.

*Demonſtr.* As CD is to G, (*a*) ſo is A to B. But the a *cor.* 19.
ratio of A to B is given; therefore alſo the ratio of CD 20. 6.
to G is given. And ſeeing that CD, EF, and G, are
proportional, (*b*) alſo the ratio of CD to EF is given. b 24. *prop.*
But A and B are given in kind: Therefore (*c*) the c 53. *prop.*
other ſides ſhall have given ratio's to the other ſides.

Now let the figure A be not alike to the figure B,
and let there be deſcribed on EF the figure EH, alike
and alike poſited to A: Therefore the figure EH is given
in kind; but the figure B is alſo given in kind: There-
fore (*d*) the ratio of B to EH is given; and therefore the d 49. *prop.*
ratio of A to the ſame EH (*e*) is alſo given; But A is e 8. *prop.*
alike to EH: Therefore (by what is aboveſaid) the ratio
of CD to EF is given; and in like manner the ratio of
the other ſides to the other ſides is given.

## OTHERWISE. *Fig.* 17.

*Conſtr.* Let there be expoſed the given line GH: Now either the figure A is alike to the figure B, or not. Let it in the firſt place be alike, and let it be as CD is to EF, ſo is GH to LK; then on GH and LK let the figures M and N be deſcribed alike, and alike poſited to the ſaid A and B, which figures M and N ſhall conſequently be given in kind.

*Demonſtr.* Therefore ſeeing that as CD is to EF, ſo is GH to LK, and that on thoſe lines CD, EF, GH, and LK, are deſcribed the figures A, B, M, and N, alike and alike poſited; (*f*) as A is to B, ſo is M to N. But the ratio of A to B is given: Therefore the ratio of M to N is given. But (*g*) M is given, conſidering that it is a figure given in kind, deſcribed on a right-line given in magnitude; therefore N is alſo given.

f 22. 6.
g 52. *prop.*

*Conſtr.* 2. Now, on LK let the ſquare O be deſcribed: Therefore (*h*) the figure O is given in kind.

h *ſch.* 52. *prop.*

*Demonſtr.* 2. Wherefore the ratio of O to N is given. But N is given: Therefore O is given; and conſequently (*i*) alſo KL. But GH is given: Therefore (*k*) the ratio of GH to KL is given. But as GH is to LK, ſo is CD to EF. Therefore the ratio of CD to EF is given; and therefore the figures A and B being given in kind, (*l*) the other ſides of the ſame figures ſhall alſo have to the other ſides given ratio's. But if the figures be not alike, the latter part of the demonſtration here above muſt be obſerved.

i *ſch.* 52. *prop.*
k 3. *prop.*
l 53 *prop.*

## PROP. LV. Plate VIII. *Fig.* 18.

*If a ſpace* A *be given in kind, and in magnitude, the ſide thereof ſhall be given in magnitude.*

*Conſtr.* For let the right-line BC, given in poſition and in magnitude, be expoſed; and thereon let there be deſcribed the ſpace D, alike and alike poſited to A; therefore the ſaid ſpace D is given in kind.

*Demonſtr.* For as it is deſcribed on the line BC, given in magnitude, it is alſo (*a*) given in magnitude. But the figure A is alſo given: Therefore (*b*) the ratio of A to D is given. But thoſe figures A and D are given in kind: Therefore (*c*) the ratio of the line EF to the line BC is given. But BC is given: Therefore (*d*) FF is alſo given. But the ratio of the ſame EF ro FG is given: Therefore (*e*) FG is given. And by the ſame ways of reaſoning it may be demonſtrated that each of the other ſides are given in magnitude.

a 52. *prop.*
b 3. *prop.*
c 54. *prop.*
d 3. *def.*
e 2. *prop.*

## OTHERWISE. *Fig.* 19.

*Conſtr.* Let the ſpace GHIKL be given in kind and in magnitude: I ſay that the ſides thereof are given in magnitude. For on the right-line GH let there be deſcribed the ſquare GM; therefore (*f*) GM is given in kind.

*Demonſtr.* But the ſpace GHIKL is alſo given in kind: Therefore (*g*) the ratio of the ſame ſpace GK to GM is given. But GK is given in magnitude: Therefore (*h*) GM is alſo given in magnitude; and ſeeing that GM is the ſquare of the line GH, (*i*) that line GH is given in magnitude. Wherefore in like manner, each of the other lines HI, IK, KL, and LG, is given.

g 49. *prop.*
h 2. *prop.*
i *ſch.* 52. *prop.*

## PROP. LVI. Plate VIII. *Fig.* 20.

*If two equiangled parallelograms* A *and* B, *have to one another a given ratio, as one ſide* CD *of the firſt* A, *is to one ſide* FG, *of the ſecond* B; *ſo the other ſide* GE, *of the ſecond* B, *is to that to which* DH, *the other ſide of the firſt* A, *hath the given ratio that the parallelogram* A *hath to the parallelogram* B.

*Conſtr.* For let HD be prolonged directly to L, ſo that as CD is to FG, ſo HD may be to DL; and finiſh the parallelogram DK.

*Demonſtr.* Seeing that as CD is to FG, ſo HD is to DL, and (*a*) that CD is equal to KL; as LK is to FG, ſo is GE to DL; and thus the ſides about the equal angles DLK and EGF are reciprocally proportional: Wherefore (*b*) DK is equal to B; and therefore ſeeing the ratio of A to B is given, and that B is equal to DK, the ratio of A to DK is given. But as (*c*) A is DK (that is to B) ſo is HD to DL: therefore the ratio of HD to DL is alſo given: and ſeeing that as CD is to FG, ſo GE is to DL, and that the right-line HD hath to DL a given ratio; to wit, that which the ſpace A hath to the ſpace B; as CD is to FG, ſo GE is to that to which HD hath the given ratio that the ſpace A hath to the ſpace B, that is to ſay, the ratio of HD to DL.

a 34. 1.
b 14. 6.
c 1. 6.

## PROP. LVII. *Fig.* 21.

*If a given ſpace* AD *be applied to a given right-line* AB *in a given angle* CAB, *the breadth* CA *of the application is given.*

*Conſtr.* For on AB, let there be deſcribed the ſquare AF; therefore (*a*) the ſame AF is given: Let the lines EA, FB, and CD, be prolonged to the points G and H.

*Demonſtr.* Seeing therefore that each ſpace AD and AF is given, their ratio is alſo given. But (*b*) AD is equal to AH: Therefore the ratio of AF to AH is given: Wherefore the ratio of EA to AG is given. (For (*c*) it

a *ſch.* 52. *prop.*
b 36. 1.
c 1. 6.

is the ſame with that of AF to AH.) But EA is equal to
AB; therefore the ratio of AB to AG is given. Now
ſeeing that the angle CAB is given, and the angle GAB
alſo given, the reſidue CAG is given. But the angle
CGA is alſo given, being a right-angle: Therefore the
remaining angle ACG is given. Wherefore the triangle
d 40. *prop.* (*d*) CAG is given in kind: Therefore the ratio of CA to
AG is given. But the ratio of AB to the ſame AG is alſo
given: Therefore the ratio of CA to AB is given; and
the ſaid AB is given: Wherefore CA is alſo given.

## PROP. LVIII. Plate VIII. *Fig.* 22.

*If a given ſpace* AB, *be applied to a given right-line* AC, *wanting by a figure* DE, *given in kind, the breadths of the defects are given.*

*Conſtr.* For let AC be divided in two equal parts in the point F: Therefore as well AF as FC is given. On the ſaid line FC let there be deſcribed the rectangled figure FG alike and alike poſited to DE. Therefore FG is given in kind.

*Demonſtr.* Seeing the figure FG is deſcribed on the
right-line FC given in magnitude, the ſaid rectiline FG
a 52. *prop.* is (*a*) alſo given in magnitude. But FG is equal to AB
b 36. 1. and IL; (for (*b*) AI and FE being equal, and (*c*) FB
c 43. 1. and BG alſo equal, the Gnomon ICL is equal to AB;
and therefore their added figure IL common to both,
FG ſhall be equal to AB and IL:) Therefore the figures
AB and IL together are given in magnitude. But AB
d 4. *prop.* is given in magnitude: Therefore (*d*) the remaining
e 24. 6. figure IL is alſo given in magnitude. But it is alſo
f 55. *prop.* given in kind, ſeeing it is (*e*) alike to DE: Therefore
g 34. 1. (*f*) the ſides of the ſame IL are given: Wherefore IB
h 4. *prop.* is given; and ſeeing that it is equal (*g*) to FD, the ſame
i 3. *def.* FD is alſo given. But FC is given; therefore the re-
k 2. *prop.* mainder DC (*h*) is given; and (*i*) in a given ratio to
BD, and therefore (*k*) BD is given.

## PROP. LIX. *Fig.* 23.

*If a given ſpace* AB *be applied according to a given right-line* AC, *exceeding it by a figure* CB *given in kind, the breadths of the exceſſes* CE *and* CF *are given.*

*Conſtr.* For DE being divided into two equal parts in G, let there be deſcribed on GE the rectiline figure GH, alike and alike poſited to CB.

*Demonſtr.* Now ſeeing that CB is alike to GH, thoſe figures CB and GH * are about one and the ſame diameter, and GH is given in kind, as is CB. But it is deſcribed on the given line GE: Therefore (*a*) the ſame GH is alſo given in magnitude. a 32. *prop.* But AB is given: Therefore AB and GH are given in magnitude. Now thoſe figures AB and GH, are equal to LI, (for AG, LE, and EI, being equal, the Gnomon GFH is equal to AB; and therefore adding GH common to both, LI ſhall be equal to AB and GH;) therefore LI is given in magnitude; but is alſo given in kind, ſince it is (*b*) alike to CB. b 24. 6. Therefore (*c*) the ſides of the ſaid LI are given, c 55. *prop.* ſeeing it is equal to GE: Therefore (*d*) the remainder CF d 4. *prop.* is given, and in a given ratio (*e*) to CE. e 3. *def.* Wherefore (*f*) CE is given. f 2. *prop.*

Scholium. Plate VIII. *Fig.* 24.

* *EUCLIDE ſuppoſeth here, that* CB *and* GH *are about one and the ſame diameter, but we ſhall thus demonſtrate it: Let* CB *and* GH *be two alike parallelograms diſpoſed as above, that is to ſay, that the equal angles join together in* E, *the ſide* CE *meets directly with his homologous ſide* EH, *and the ſide* BE, *his correſpondent ſide* EG; *and let the diameter* FE *be drawn, I ſay that the ſaid diameter* FE *prolonged, will paſs through the point* K; *that is to ſay, the parallelograms* GH *and* CB, *conſiſt about one and the ſame diameter. For if it be denied, the diameter* EF *being produced, will paſs above the point* K, *or below it. Let it in the firſt place paſs above it, and let it cut* GK, *prolonged in the point* M, *and through the point* M *let there be drawn* MN, *parallel to* KH, *which ſhall meet* EH, *prolonged in the point N, and* FB *in O.*

*Demonſtr.* Foraſmuch as the parallelograms GN and CB are with the parallelogram LO about one and the ſame diameter, they are (*g*) alike to one another. g 24. 6. Wherefore as FC is to CE, ſo is EG to GM. In like manner, ſeeing the parallelograms CB and GH are alike, as FC is to CE, ſo is EG to GK: Therefore (*h*) as EG is to GM, ſo is EG to GK. h 11. 5. Wherefore (*i*) GM and GK are equal, a part to the whole, which is abſurd: i 9. 5. By the ſame way of reaſoning it may be demonſtrated. that the diameter prolonged will not fall below the point K: Therefore the parallelograms CB and GE conſiſt about one and the ſame diameter.

## PROP. LX. Plate VIII. *Fig.* 25.

*If a parallelogram* AB, *given in kind and in magnitude, be augmented or diminiſhed by a Gnomon* CFD, *the breadth of the Gnomon (conſiſting of the lines* CE *and* DG) *are given.*

*Demonſtr.* For ſeeing that AB is given, and the Gnomon CFD alſo given, the whole parallelogram BF is given: But it is alſo given in kind, ſeeing it is alike to BA: Therefore (*a*) the ſides of the ſame BF are given; a 55. *prop.* and therefore each of the lines BE and BG is given. But each of the lines BC and BD is given; therefore each of the remaining lines CE and DG is alſo given.

*Conſtr.* Now let the parallelogram BF, given in kind and in magnitude, be diminiſhed by the given Gnomon CFD: I ſay that each of the lines CE and DG is given.

*Demonſtr.* For ſeeing that BF is given, and the Gnomon CFD given, the remaining figure AB is alſo given. But it is alſo given in kind, ſeeing it is alike to BF: Therefore (*b*) the ſides of the ſaid AB are given, and there- b 55. *prop* fore each of the lines CB and BD is given, but each of the lines BE and BG is given: Therefore alſo each of the remaining lines CE and DG is given.

## PROP. LXI. *Fig.* 26.

*If to one ſide of a figure* ABCE, *given in kind, there be applied a parallelogrammic ſpace* CD *in a given angle* BCF, *and that the given figure* AC *hath to the parallelogram* CD *a given ratio, the parallelogram* CD *is given in kind.*

*Conſtr.* For through the point B, let BH be drawn parallel to CE, and through the point E let EH be drawn parallel to CB, and let EC and HB be prolonged to the points K and G.

*Demonſtr.* Foraſmuch as the angle BCE is given, a 3. *def.* and the ratio of EC to CB, (*a*) the parallelogram CH is given * in kind. But the figure ABCE is alſo given in kind, and is deſcribed on the ſame line BC, as the b 49. *prop.* parallelogram CH given in kind is: Therefore (*b*) the ratio of the figure ABCE to the parallelograms CH is given. But by ſuppoſition the ratio of the ſaid figure ABCE to the parallelogram CD is alſo given; and CD c 36. 1. is (*c*) equal to CG: therefore (*d*) the ratio of CH to d 8. *prop.* CG is given. Wherefore the ratio of the line EC to e 1. 6. the line CK is given; (for (*e*) as CH is to CG, ſo is EC to CK.) But the ratio of EC to CB is alſo given: There- f 8 *prop.* fore (*f*) the ratio of the ſaid CB to CK is given. And ſeeing that the angle ECB is given, alſo the following angle BCK (*g*) is given. But the angle BCF is propoſed given; and therefore the remaining angle FCK is given. g 13. 1 & 4. *prop.* Alſo the angle CKF is given, for that (*h*) it is equal to the angle BCK: h 29. 1. Therefore the other angle CFK is given: Wherefore (*i*) the triangle FCK is given in kind; i 40. *prop.* and therefore the ratio of FC to CK is given. But the ratio of CB to the ſame CK is alſo given. Therefore (*k*) the ratio of FC to CB is given; k 8. *prop.* and the angle BCF is alſo given. Wherefore the parallelogram CD is given in kind.

### Scholium.

* *Altho' it be manifeſt that a parallelogram that hath one angle given, and the ratio of the ſides about the ſame angle alſo given, is given in kind, as* Euclide *declares, yet the antient interpreter thus demonſtrates it.*

*Seeing that in the parallelogram* CH *the angle* ECB *is given, the angle* CEH *is alſo given; for the right-line* EC *falling on the parallels* EH *and* CB, *doth make the two internal angles on the ſame part equal to two right-angles. And therefore ſeeing that the angle* ECB *is given, the other angles are given; and ſeeing that the ratio of* EC *to* CB *is given, and that* BH *is equal to* CE, *and* EH *to* BC, *the ratio of the ſides to one another is alſo given.*

## PROP. LXII. Plate VIII. *Fig.* 11, 12.

*If two right-lines* AB *and* CD, *have to one another a given ratio, and that on one of them* AB, *there be deſcribed a figure* AEB, *given in kind; but on the other* CD, *a parallelogrammic ſpace* DF *in a given angle* DCF, *and that the figure* AEB *hath to the parallelogram* DF *a given ratio, the parallelogram* DF *is given in kind.*

*Conſtr.* For on the line AB let there be deſcribed the parallelogram AH, alike and alike poſited to DF.

*Demonſtr.* Seeing that the ratio of AB to CD is given, and that on thoſe lines are deſcribed the rectiline figures AH and FD, alike and alike poſited, (*a*) the ratio of AH a 50. *prop.* to FD is given. But the ratio of FD to AEB is alſo given: Therefore (*b*) the ratio of AH to AEB is given. b 8. *prop.* But the angle ABH is alſo given, being equal to the angle FCD, and ſo the figure AEB is given in kind; and to AB one of the ſides thereof, the parallelogram

AH is applied in a given angle ABH, and the ratio of
the ſaid figure AEB to the ſaid parallelogram AH is gi-
c 61. *prop.* ven: Therefore (*c*) the parallelogram AH is given in
kind; and therefore FD which is alike thereto, is alſo
given in kind.

## PROP. LXIII. Plate VIII. *Fig.* 27.

*If a triangle* ABC *be given in kind, the ſquares* BE, CD, *and* CF, *which is deſcribed on each of the ſides, ſhall have a given ratio to the triangle* ABC.

*Demonſtr.* For ſeeing that on one and the ſame right-
line BC, there are deſcribed the two rectiline figures
a 49. *prop.* ABC and CD, given in kind, (*a*) the ratio of the ſame
ABC to CD is given; and therefore the ratio of the
ſquares BE and CF, to the triangle ABC is alſo given.

## PROR. LXIV. *Fig.* 5.

*If a triangle* ABC, *hath an obtuſe angle* ABC *given, that ſpace by which the ſide* AC *ſubtending the obtuſe angle* ABC, *is more in power than the ſides* AB *and* BC, *that comprehend the ſaid angle, ſhall have a given ratio to the triangle* ABC.

*Conſtr.* Let the line CB be prolonged directly, and
from the point A let the perpendicular AD be drawn:
I ſay that the ſpace by which the ſquare of the line AC
doth exceed the ſquares of the lines AB and BC, that is
a 12. 2. to ſay, (*a*) the double of the rectangle contained under
CB and BD, ſhall have a given ratio to the triangle
ABC.

*Demonſtr.* For ſeeing that the angle ABC is given,
the angle ABD is alſo given; but the angle ADB
is alſo given; therefore the other angle BAD is
b 40. *prop.* given: Wherefore (*b*) the triangle ABD is given in
c 3. *def.* kind; therefore (*c*) the ratio of AD to DB is given.
d 1. 6. But as AD to DB, ſo (*d*) the rectangle of AD and BC is
to the rectangle of BC and BD. But the ratio of AD to
BD is given: Therefore alſo is the ratio of the rectangle
of AD and BC to the rectangle of BC and BD given:
Wherefore the ratio of the double of the ſaid rectangle
BC and BD to the rectangle of AD and BC is alſo given.
But the ſaid rectangle of AD and BC hath alſo a given
ratio to the triangle ABC (to wit, a double ratio; for
e 41. 1. the rectangle is (*e*) double to the triangle) therefore the
f 8. *prop.* ratio of the double of the rectangle of BC and BD (*f*)

Plate VII. *Facing Pag. 294.*

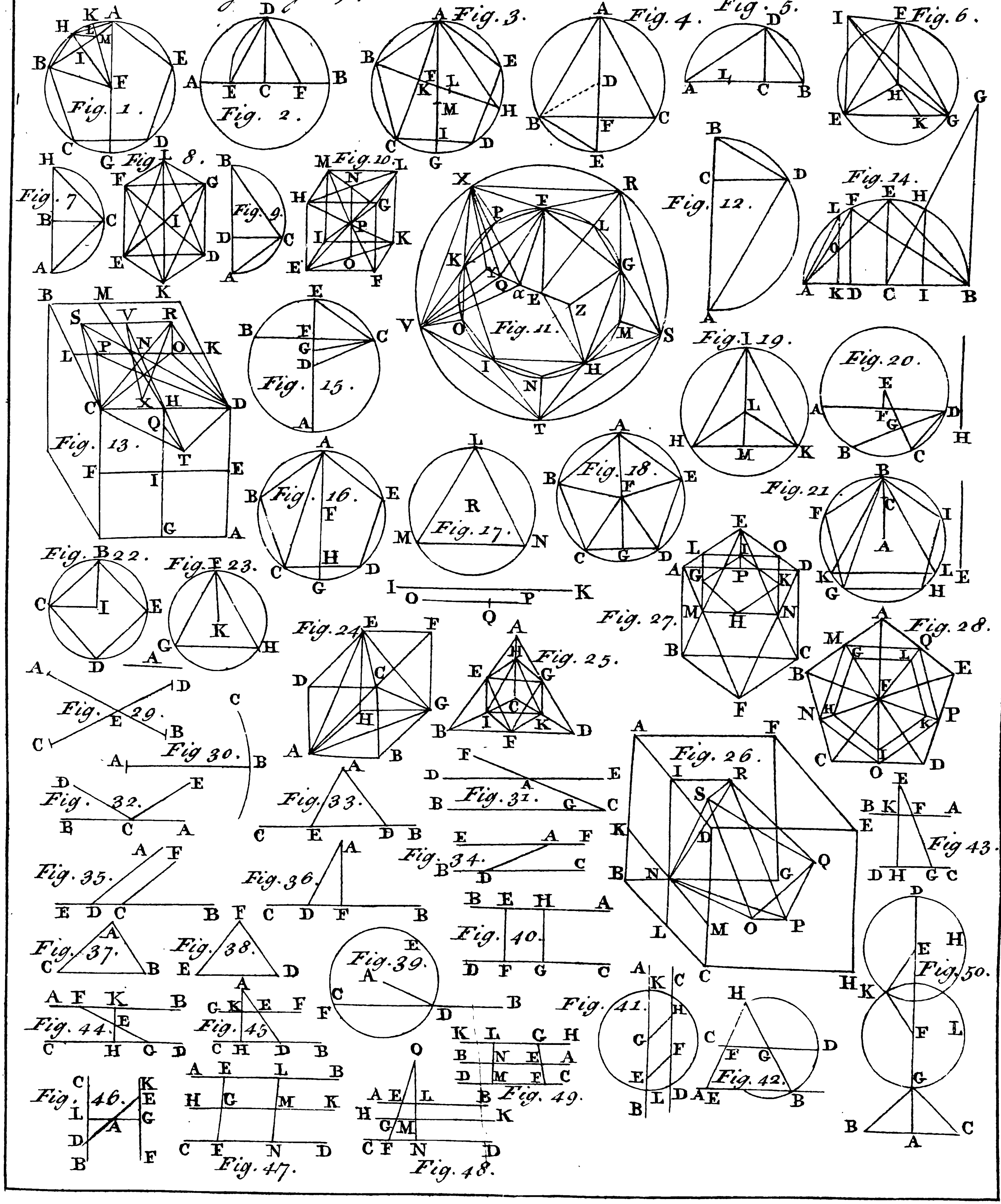


Fig. 1.
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Fig. 47.
Fig. 48.
Fig. 49.
Fig. 50.

to the triangle ABC is given. But the ſame double of the rectangle of CB and BD is that ſpace by which the ſquare of the line AC doth exceed the ſquares of the lines AB and BC: Therefore the ſame ſpace hath a given ratio to the triangle ABC.

### PROP. LXV. Plate VIII. *Fig.* 4.

*If a triangle* ABC, *hath one acute angle* ACB *given, that ſpace, by which the ſide ſubtending the ſaid acute angle is leſs in power than the ſides comprehending the ſame acute angle, ſhall have a given ratio to the triangle.*

*Conſtr.* From the point A let there be drawn the line AD, perpendicular to BC: I ſay, that ſpace by which the ſquare of the line AB is leſs than the ſquares of the lines AC and CB, that is to ſay, (*a*) the double of the rectangle of BC and CD, hath a given ratio to the triangle ABC. a 13. 2.

*Demonſtr.* For ſeeing that the angle C is given, and the angle ADC alſo given, the other angle DAC is given: Wherefore the triangle (*b*) ADC is given in kind; and therefore the ratio of AD to DC is given, and conſequently alſo (*c*) that of the rectangle of BC and CD to the rectangle of BC and AD: Therefore the ratio of the double of the rectangle of BC and CD to the rectangle of BC and AD is given. But the ratio of the ſame rectangle of BC and AD to the triangle ABC is given (for (*d*) the rectangle is double to the triangle:) Therefore (*e*) the ratio of the double of the rectangle of BC and CD to the triangle ABC is given. And ſeeing that the ſame double of the rectangle of BC and CD is that whereby the ſquare of the line AB is leſs than the ſquares of the lines AC and BC, that ſpace by which the ſquare of the line AB is leſs than the ſquares of the lines AC and BC, ſhall have a given ratio to the triangle ABC. b 40. *prop.* c 1. 6. d 41. 1. e 8. *prop.*

### PROP. LXVI. *Fig.* 4.

*If a triangle* ACB, *hath one angle* B *given, the rectangle made of the lines* AB *and* BC, *containing the ſame angle, ſhall have a given ratio in the triangle.*

*Conſtr.* For from the point A let AD be drawn perpendicular to CB.

*Demonſtr.* Therefore ſeeing that the angle B is given, and alſo the angle ADB; the other angle BAD is likewiſe given. Wherefore the triangle ADB (*a*) is given in a 40. *prop.*

in kind; and conſequently the ratio of AB to AD is given. b 1.6. But as AB is to AD, (*b*) ſo the rectangle of AC and CB is to the rectangle of CB and AD: Therefore the ratio of the rectangle of AC and CB to the rectangle of CB and AD is given. But the ratio of the ſaid rectangle of CB and AD to the triangle ACB is alſo given; (for that it is double ratio, the rectangle c 41. *prop.* being double (*c*) to the triangle:) Therefore (*d*) the ratio d 8. *prop.* of the rectangle of AC and CB to the triangle ABD is given.

## PROP. LXVII. Plate VIII. *Fig* 28.

*If a triangle* ABC *hath one angle* BAC *given, that ſpace by which the ſquare of the line compounded of the two ſides* BA *and* AC, *that contain the ſame given angle* BAC *doth exceed the ſquare of the other ſide, ſhall have a given ratio to the triangle* ABC.

*Conſtr.* For let BA be prolonged in ſuch ſort as that AD may be equal to AC, then having drawn DCE infinitely, from the point B let BE be drawn parallel to AC, meeting the ſaid DE in the point E.

a 4. 6. & 14. 5. *Demonſtr.* Foraſmuch as AD is equal to AC, (*a*) DB is equal to BE; (for the two triangles ADC and BDE are alike) and from the top B is drawn to the baſe DE, the right-line BC: Therefore * the rectangle of DC and CE, with the ſquare of BC, is equal to the ſquare of BD; but the ſame BD is compounded of BA and AC; therefore the ſquare of the compound of AB and AC is greater than the ſquare of BC, of the rectangle of DC and CE.

Now I ſay that the rectangle of DC and CE hath a given ratio to the triangle ABC: Foraſmuch as the angle BAC is given, the angle DAC is alſo given. But each of the angle ADC and ACD is given, it being the half b 40. *prop.* of the angles BAC which is given. Therefore (*b*) the triangle ADC is given in kind; and therefore the ratio c 50. *prop.* of DA to DC is given. Therefore (*c*) the ratio of the ſquare of the ſaid DA to the ſquare of DC is alſo given. d 2. 6. And ſeeing that as BA is to AD, (*d*) ſo is EC to CD, e 1. 6. and alſo as BA is to AD, (*e*) ſo is the rectangle of BA f 1. 6. and AD to the ſquare of AD; and as EC is to CD, (*f*) ſo alſo is the rectangle of EC and CD to the ſquare of CD; by permutation, as the rectangle of BA and AD is to the rectangle of EC and CD, ſo is the ſquare of AD to the ſquare of DC. But the ratio of the ſaid ſquare of AD to the ſquare of DC is given: Therefore the ratio of the rectangle of BA and AD to the rectangle of EC and CD is alſo given. But AD is equal to AC: Therefore the ratio of the rectangle of BA and AC to the rectangle of EC and CD is given. But the ratio of the rectangle of BA and AC to the triangle ABC *(g)* is given, becauſe the angle BAC is given: Therefore *(h)* the ratio of the rectangle EC and CD to the triangle ABC is given. But the rectangle of EC and CD is that whereof the ſquare of the line compounded of BA and AC is greater than the ſquare of BC: Therefore that ſpace by which the ſquare of the line compounded of BA and AC is greater than the ſquare of BC, ſhall have a given ratio to the triangle ABC. g 66. *prop.* h 8. *prop.*

## Scholium.

* *EUCLIDE ſuppoſeth in this place, that when in an Iſoſceles triangle a right-line is drawn from the top to the baſe, the ſquare of that line, with the rectangle contained under the ſegments of the baſes, is equal to the ſquare of either of the other legs, which the antient interpreter doth thus demonſtrate.*

*Conſtr.* Let ABC be an Iſoſceles triangle, whoſe legs are AB and AC; and from the top A let AD be drawn to the baſe BC: I ſay, that the ſquare of AD with the rectangle of BD and DC, is equal to the ſquare of either of the legs AB or AC.

*Demonſtr.* Now the line AD (*Fig.* 4.) is perpendicular to BD, or not: Let it in the firſt place be perpendicular: Therefore it will cut the baſe BC into two equal parts in the point D; and therefore the rectangle contained under BD and DC is equal to the ſquare of the ſaid BD, and adding to them the common ſquare of AD, the rectangle of BD and DC with the ſquare of AD, ſhall be equal to the ſquares DB and AD. But to thoſe ſquares of AD and DB *(i)* the ſquare AB is equal: Therefore the ſquare of AB is equal to the rectangle of BD and DC, and the ſquare of AD together. i 47. 1.

Now ſuppoſe AD not to be perpendicular, but that from the point A there doth fall on BC the perpendicular AE, that being ſo, BC ſhall be cut into two parts equally in the point E, and unequally in D. Wherefore the rectangle of BD and DC, with the ſquare of DE, *(k)* is equal to the ſquare of BE; and adding the common ſquare of AE, the rectangle of BD and DC, with the ſquares of DE and AE, ſhall be equal to the ſquares of BE k 5. 2.

l 47. 1. BE and AE. But (*l*) the ſquare AD is equal to the two
ſquares of DE and AE: Therefore the rectangle of BD
and DC, with the ſquare of AD, is equal to the ſquares
of BE and AE. But to theſe ſquares of BE and AE
the ſquare of AB is equal: Therefore the ſquare of AD,
with the rectangle of BD and DC, is equal to the ſquare
of AB.

## OTHERWISE.

*Conſtr.* Having done, as in the foregoing Demonſtration, from the point A, (*Fig.* 29.) let AF be drawn perpendicular to CD, and let AE be drawn.

*Demonſtr.* Foraſmuch as the angle BAC is given, the
half thereof ACF ſhall be alſo given. But the angle
AFC is given; and therefore the triangle AFC is given
in kind: Therefore the ratio of AF to FC is given.
But the ratio of CD to the ſame FC is alſo given, ſeeing
m 8. *prop.* that CD is double to FC: Therefore (*m*) the ratio of CD
to AF is given; and therefore alſo the ratio of the
rectangle of CD and EC, to the rectangle of AF and
n 1. 6. EC, is given; (for it is the ſame ratio (*n*) as that of CD
to AF.) But the ratio of the rectangle of AF and FC
o 41. 1. to the triangle ACE is given; ſeeing it is double (*o*) to
the ſame triangle. Therefore the ratio of the rectangle
of CD and CE to the triangle ACE is alſo given. But
p 37. 1. the triangle ACE is equal to the triangle ABC (*p*), they
being both conſtituted on one and the ſame baſe AC,
and between the ſame parallels AC and BE: Therefore
q 8. *prop.* (*q*) the ratio of the rectangle of CE and CD to the
triangle ABC is given. But the ſaid rectangle of CE
and CD is the ſpace by which the ſquare of the line
compounded of AB and AC, is greater than the ſquare
of BC: Therefore that ſpace by which the ſquare of
the line compounded of AB and AC is greater than the
ſquare of BC, hath a given ratio to the triangle ABC.

## OTHERWISE. *Fig.* 4.

For the given angle A is either a right, acute, or
obtuſe angle: Let it in the firſt place be ſuppoſed a
right-angle: Therefore the ſquare of the line compounded of BAC, is greater than the ſquare of BC, by
r 47. 1. twice the rectangle of BA and AC; (ſeeing that (*r*) the
ſquare of BC is equal to the ſquares of BA and AC;
s 5. 2, and the ſquare of the line compounded of BAC (*s*) is
equal

equal to thoſe two ſquares of BA and AC, and twice the rectangle of the ſaid BA and AC :) Wherefore the ratio of double the rectangle of BA and AC to the triangle ABC is given.

*Conſtr.* Now let the angle C (*Fig.* 4.) be ſuppoſed acute, and from the point A let there be drawn on CB the perpendicular AD.

*Demonſtr.* Foraſmuch as the triangle CAB is an Oxigonium triangle, and the perpendicular AD being drawn, the ſquares of CA and CB are equal (*t*) to the ſquare of AB with twice the rectangle of CB and CD; adding therefore the common double rectangle of CA and CB, the ſquares of CA and CB, with the double rectangle of the ſaid CA and CB, that is to ſay, (*u*) the ſquare of the line compounded of ACB, are equal to the ſquare of AB, with the double of the rectangle of CD and CB, and over and above the double of the rectangle of AC and CB, that is to ſay, the double of the rectangle contained under the compound-line of ACD and CB (for the rectangle of ACD and CB is (*x*) equal to the rectangles of AC and CB, and of CD and CB :) Therefore the ſquare of the line compounded of ACB is greater than the ſquare of AC, by double the rectangle of ACD and CB. And ſeeing that the angle ACB is given, and the angle BDA alſo given, the other angle CAD is given: Therefore (*y*) the triangle CAD is given in kind, and therefore the ratio of CD to CA is given, and by conſequence the ratio of the line compounded of ACD to CA (*z*) is alſo given. Wherefore the ratio of the rectangle of thoſe lines compounded of ACD and CB (*a*) to the rectangle of AC and CB is alſo given. But the ratio of the ſaid rectangle of AC and CB to the triangle CAB (*b*) is given, ſeeing the angle C is given; therefore the ratio of double the rectangle of the line compounded of ACD and CB to the triangle CAB is given.

t 13. 5.
u 4. 2.
x 1. 2.
y 40. *prop.*
z 6. *prop.*
a 1. 6.
b 66. *prop.*

Laſtly, let the angle BAC (*Fig.* 30.) be ſuppoſed to be obtuſe, and having prolonged BA, from the point C, let the perpendicular CE be drawn on the ſaid line BA prolonged; and let AF be propoſed to be equal to AE.

*Demonſtr.* Foraſmuch as the angle BAC is obtuſe, and the perpendicular CE being drawn, the ſquares of AB and AC, and the double of the rectangle under BA and AE, or AF, are all alike equal (*c*) to the ſquare of BC, and adding the common double rectangle of BA and AC, the ſquares of the ſaid AB and AC, with the double of the

c 12. 2.

the rectangle of the ſame AB and AC, that is to ſay,
d 4. 2. (*d*) the ſquare of the line compounded of BAC and the
double of the rectangle of BA and AF are together
equal to the ſquare of BC, with the double of the rect-
angle of BA and AC. Let the common double of the rect-
angle of BA and AF be taken away, and there will re-
main the ſquare of the line compounded of BAC, equal
to the ſquare of BC, with the rectangle of AB and CF;
e 1. 2. (for the rectangle of AB and AC is equal (*e*) to the two
rectangles of AB and AE, and of AB and CF:) There-
fore the ſquare of the line compounded of BAC is greater
than the ſquare of BC by the double of the rectangle
of AB and CF. And foraſmuch as the angle BAC is
f 13. 11. given, the angle CAE (*f*) is given. But the angle AEC
is alſo given; therefore the other angle ACE is given:
g 40. *prop.* Wherefore (*g*) the triangle ACE is given in kind, and
therefore the ratio of CA to AE, that is to ſay, to AF
h 5. *prop.* is given. Therefore (*h*) the ratio of the ſaid CA to FG is
alſo given. But the ratio of the ſame CA to CE is
i 8. *prop.* given; therefore (*i*) the ratio of CE to CF is alſo given.
Wherefore the ratio of the rectangle of EC and AB to
the rectangle of FC and AB is given; (for the rectangle
k 2. 6. is to the rectangle (*k*) as CE is to CF) and alſo that of the
rectangle of AC and AB to the rectangle of EC and
l 8. *prop.* AB. Therefore (*l*) the ratio of the rectangle of FC and
AB to the rectangle of AC and AB is given. But the
ratio of the rectangle of AC and AB to the triangle ABC
m 66. *prop.* (*m*) is given: Therefore alſo the ratio of the double of
the rectangle of FC and AB, to the triangle ABC is given.
But the ſame double of the rectangle of FC and AB, is
that whereby the ſquare of the line compounded of BAC
is greater than the ſquare of BC, wherefore that ſpace
by which the ſquare of the line compounded of BAC is
greater than the ſquare of BC, hath a given ratio to the
triangle ABC.

### OTHERWISE. Plate VIII. *Fig.* 31.

*Conſtr.* Let the line BA be prolonged to the point D, in ſuch ſort as AD may be equal to AC, and let CD be drawn.

*Demonſtr.* Foraſmuch as the angle BAC is given, each
of the angles ADC and ACD, which is the half thereof,
ſhall be alſo given; and therefore the other angle DAC
n 40. *prop.* is alſo given: Therefore (*n*) the triangle ACD is given in
kind. Wherefore the ratio of AC to CD is given. And
foraſmuch as the angle ADC is given: Let each of the
angles

angles DEC and AFC be made † equal to the ſaid
ADC: Therefore ſeeing that the angle BDC is equal
to the angle DEC, and the angle DBE is common to
the triangles DBE and DBC, the other angle BDE is
equal to the other angle BCD; and therefore the trian-
gle BDE is equiangled to the triangle BDC. Therefore
(*o*) as EB is to BD, ſo is BD to CB: Wherefore the rect- o 4. 6.
angle of EB and CB, that is to ſay, (*p*) the rectangle of p 5. 2.
EC and CB, (*q*) with the ſquare of CB is equal, (*r*) to q 5. 2.
the ſquare of BD, that is to ſay, to the ſquare of the line r 17. 6.
compounded of BAC; for AD is equal to AC; and
therefore the rectangle of EC and CB with the ſquare
of CB, that is to ſay, the ſquare of the line compounded
of BAC is greater than the ſquare of the rectangle of
BC and CE: I ſay therefore that the ratio of the ſaid
rectangle of BC and CE to the triangle ABC is given.
Foraſmuch as the angle BDE is equal to the angle
BCD, and the angle ADC equal to the angle ACD,
the other angle CDE is equal to the other angle ACB:
But the angle DEC is equal to the angle AFC; therefore
the remaining angle CAF is equal to the remaining
angle DCE. Wherefore the triangle AFC is equian-
gled to the triangle DCE; and therefore (*s*) as CA is s 4. 6.
to AF, ſo is CD to CE; and by permutation, as AC is
to CD, ſo is AF to CE. But the ratio of AC to CD
is given: Therefore alſo the ratio of AF to CE is given.
From the point A let AH be drawn perpendicular to BC:
Foraſmuch as the angle AFC is given, and the angle
AHF alſo given, the third angle HAF is given:
Wherefore (*t*) the triangle AHF is given in kind; and t 40. *prop.*
by conſequence the ratio of AF to AH is given. But
the ratio of AF to CE is alſo given: Therefore (*u*) u 8. *prop.*
the ratio of AH to CE is given; and therefore the
ratio of the rectangle of AH and BC (*x*) to the rectangle x 1. 6.
of BC and CE is alſo given. But the ratio of the
rectangle of AH and BC, to the triangle ABC is like-
wiſe given; (for the rectangle (*y*) is double to the tri- y 41. 1.
angle) and the rectangle of BC and CE is that whereby
the ſquare of the line compounded of BAC is greater
than the ſquare of BC. Therefore that ſpace by which
the ſquare of the line compounded of BAC is greater
than the ſquare of BC has a given ratio to the triangle
ADC.

### Scholium.

† *The antient Interpreter pretending to ſhew the conſtruction of the angle* DEB *equal to the angle* ADC, *ſaith that on the line* BD *and in the point* D, *the angle* BDE *ought be made equal to the angle* BCD, *and that the right-lines* BC *and* DE *be drawn until they interſect in* E, *in ſuch ſort as he ſuppoſeth the angle* BCD, *to be given, but it is not.*

*The ſame Interpreter afterwards ſhews how there may univerſally from a given point be drawn a right-line, given in poſition to a right-line, making an angle equal to a given angle: But we will alſo reject this way, ſeeing we have elſewhere ſhewn another more brief and eaſy. For example, if we would from the point* D *draw to the line* BC *given in poſition a right-line, making an angle equal to a given angle* ADC, *as is here required, we have no more to do but to aſſume the point* K *in the ſaid line* BC, *and there make the triangle* CKL *equal to the given angle* ADC : *If the line* KL *doth meet with the point* D, *it ſhall be the line required. But if it meet not with it, from the point* D *let there be drawn the line* DE *parallel to the ſaid* KL, *cutting* BC *prolonged in* E, *and the angle* DEC *ſhall be equal to the given angle* ADC, *for on the two parallel-lines,* LK *and* DE, *there doth fall the* z 29. 1. *line* BE ; *and therefore the angle* DEC (z) *is equal to the angle* LKC, *which hath been made equal to the given angle* ADC ; *and by conſequence the ſame angle* DEC *is alſo equal to* ADC,

### PROP. LXVIII. Plate VIII. *Fig.* 32, 33.

*If two parallelograms* AB *and* CD *have to one another a given ratio, and that a ſide hath alſo a given ratio to a ſide, the other ſide ſhall have likewiſe a given ratio to the other ſide.*

*Conſtr.* Let the ratio of BE to FD be given: I ſay the ratio of AE to FC is alſo given. For to the right-
a 1. 6. line EB let there be applied the parallelogram EH, equal to the parallelogram CD, and conſtituted in ſuch ſort
b 14. 6. as AE and EG may make one right-line: † Therefore KB and BH will alſo make one right-line.

*Demonſtr.* Foraſmuch as the ratio of AB to CD is given, and that EH is equal to the ſaid CD; the ratio of (*a*) AB to EH is given; and therefore the ratio of AE to EG is alſo given. Seeing therefore that EH is equal and equiangled to CD, as (*b*) EB is to FD, ſo is FC to EG. But the ratio of EB to FD is given: Therefore alſo the ratio of FC to EG is given. But the ratio of AE to the ſame EG is alſo given: Therefore the ratio of AE to FC is given.

### Scholium.

† *EUCLIDE having poſited* AE *and* EG *directly in one right-line, preſently concludes that* KB *and* BH *ſhall alſo make a right-line; but we ſhall demonſtrate it thus. Seeing the lines* AE *and* EG *are poſited directly, the angles* AEB *and* BEG (c) *are equal to two right-angles; and ſeeing* c 13. 1. *that* AB *is a parallelogram, the lines* AK *and* EB *are parallels, on which the line* AE *doth fall; and therefore the two internal angles* A *and* BEA (d) *are alſo equal to* d 29. 1. *two right-angles, and taking away the common angle* BEA, *there will remain the angle* A, *equal to the angle* BEG; *and conſequently their oppoſite angles* EBK *and* H *are alſo equal to one another. Again, ſeeing that* BG *is a parallelogram, the two lines* BE *and* HG *are parallels, on which* BH *falls; and therefore the two internal angles* H *and* EBH *are equal to two right-angles. But it hath been demonſtrated that* H *is equal to* EBK: *Therefore the two angles* EBK *and* EBH *are alſo equal to two right-angles; and therefore* (e) *the two lines* KB *and* BH *meet directly according to EUCLIDE.* e 14. 1.

### OTHERWISE, Plate VIII. *Fig.* 34, 35.

*Conſtr.* Let the given right-line K be expoſed, and ſeeing that the ratio of A to B is given, let the ſame be made of K to L; therefore the ratio of K to L is alſo given.

*Demonſtr.* But K is given; therefore (*f*) L is alſo given. f 2. *prop.* Again, ſeeing that the ratio of CD to EF is given, let the ſame be made of K to M: Therefore the ratio of K to M is given. But K is given, therefore (*g*) M is alſo g 1. *prop.* given; and therefore the ratio of L to M is given. Now ſeeing that A is equiangled to B, (*h*) the ratio h 23. 6. of the ſaid A to B is compounded of that of the ſides, that is to ſay, CD to EF, and of CG to EH. But alſo the ratio of K to L is compounded of K to M, and of M to L; therefore the ratio compounded of CD to EF, and of CG to EH, is the ſame with that which is compounded of K to M, and of M to L (the ratio of K to L being the ſame as of A to B :) But the ratio of CD to EF is the ſame as of K to M : Therefore the other ratio of CG to EH is alſo the ſame as of M to L. But the ſaid ratio of M to L is given : Therefore alſo the ratio of CG to EH is given.

### PROP. LXIX. Plate VIII. *Fig.* 36, 37.

*If two parallelograms,* CB *and* EH, *having the angles* D *and* F *given, and that a ſide hath alſo a given ratio to a ſide ; in like manner the other ſide ſhall have a given ratio to the other ſide.*

*Conſtr.* Let the ratio of BD to FH be alſo given : I ſay that the ratio of AB to EF is given. For if CB be equiangled to HE, it is manifeſt by the precedent Propoſition ; but if it be not equiangled thereto, let the right-line DB be conſtituted, and in the given point B therein, let the angle DBK be made equal to the angle EFH, and finiſh the parallelogram DK.

*Demonſtr.* Foraſmuch as each of the angles BKL and BAK is given, † the other angle KBA is given : Wherefore the triangle (*a*) ABK is given in kind ; and therefore the ratio of AB to BK is given. But the ratio of CB to EH is ſuppoſed to be given, and (*b*) CB is equal to DK ; therefore the ratio of DK to EH is given ; and ſeeing that DK is equiangled to EH, and the ratio of the ſaid DK to EH is given, as alſo that of DB to FH, (*c*) the ratio of BK to FE is given. But the ratio of the ſaid BK to BA is alſo given : Therefore (*d*) the ratio of AB to FE is given.

a 40. *prop.*
b 35. *prop.*
c 68. *prop.*
d 29. 1.

Scholium.

† *EUCLIDE ſuppoſeth here, that a parallelogram having one angle given, all the other angles are alſo given, and as well the antient Interpreters as others give the reaſons why, the angle* F *being given, the other angle* E *ſhall be alſo given, it being the remainder of two right-angles, for that on the parallel-lines* EG *and* FH *there falls the line* EF, *which makes* (e) *the two internal angles* (*of the ſame part*) F *and* G, *equal to two right-angles. But to thoſe angles* (f) *the oppoſite angles* G *and* H *are equal, and therefore they are alſo given.*

e 29. 1.
f 34. 1.

*From whence it follows that the angles* BDC *and* F *being given by ſuppoſition, all the other angles of the two parallelograms* CB *and* EH, *are alſo given : Therefore the angle* DBK *having been made equal to the angle* F, *the angle* K *ſhall be equal to the angle* E, *and given as that* *is; But the angle* BAL, *which is oppoſite to the given angle* BDC, *is alſo given; and therefore* BAK, *which is the remainder of two right-angles, ſhall be alſo given; in ſuch ſort as in the triangle* ABK, *the two angles* BAK *and* BKA *are given, as EUCLIDE declares in this place.*

## PROP. LXX. Plate VIII. *Fig.* 38, 39.

*If of two parallelograms* AB *and* EH, *the ſides about the equal angles, or about the unequal angles (yet nevertheleſs given angles) have to one another a given ratio, to wit,* (AC *to* EF, *and* CB *to* FH) *alſo the ſame parallelograms* AB *and* EH *ſhall have to one another a given ratio.*

*Conſtr.* For let AB be prolonged to EH, and on the right-line CB let the parallelogram CM be applied equal to the parallelogram EH, in ſuch ſort as AC may be directly to CN; that is to ſay, that AC and CN make one right-line; and by conſequence DB ſhall be (*a*) directly with BM. a *ſch.* 68. *prop,*

*Demonſtr.* Foraſmuch then as CM is equiangled and equal to EH, the ſides about the equal angles ſhall be reciprocally (*b*) porportional: Wherefore as BC is to HF, b 14. 6. ſo is FE to NC. But the ratio of BC to HF is given,

Therefore the ratio of FE to NC is alſo given. But the ratio of AC to the ſame EF is given: Therefore (*c*) the ratio of AC to NC is alſo given. c 8. *prop.* Wherefore the ratio of AB to CM is given: (for it is, the ſame (*d*) as of AC to CN.) But CM is equal to EH: d 1. 6 Therefore the ratio of AB to EH is given.

*Conſtr.* Now ſuppoſe AB not to be equiangled to EH, and on the right-line CB, and in the given point C therein, let there be conſtituted the angle BCK, equal to the given angle F, and ſo finiſh the parallelogram CL.

*Demonſtr.* Foraſmuch as the angle ACB is given, and the angle BCK alſo given, the remaining angle ACK is given: Therefore the triangle ACK (*e*) is given e 40. *prop.* in kind: and therefore the ratio of AC to CK is given: But the ratio of AC to EF is alſo given; Therefore the ratio of CK to EF is given, But the ratio of BC to HF is alſo given, and the angle BCK is equal to the angle F; therefore (by the firſt part of this propoſition) the ratio of CL to EH is given. But to the ſaid CL, AB is equal: Therefore the ratio of AB to EH is given.

## PROP. LXXI. Plate VIII. *Fig.* 40, 41.

*If of two triangles* ABC *and* DEF, *the ſides about the equal angles* A *and* D, *or elſe about the unequal angles (yet neverthleſs given angles) have to one another a given ratio (to wit,* AB *to* DE, *and* AC *to* DF) *the ſame triangles ſhall have alſo to one another a given ratio* ABC *to* DEF.

*Conſtr.* Let the parallelograms AG and DH be finiſhed.

*Demonſtr.* Seeing that the two parallelograms AG and DH, have the ſides about the equal angles A and D, or elſe about the unequal angles (neverthleſs given) in a given ratio to one another, the ratio (*a*) of the parallelogram AG to the parallelogram DH is given. But the triangle ABC is the half of the parallelogram AG (*b*) and the triangle DEF the half of the parallelogram DH. Therefore the ratio of the triangle ABC to the triangle DEF is given.

a 70. *prop.*

b 34. *prop.*

## PROP. LXXII. *Fig.* 42, 43.

*If of two triangles* ABC *and* DEF, *the baſes* BC *and* EF, *are in a given ratio,* BC *to* EF, *and that from the angles* A *and* D, *there be drawn to thoſe baſes the right-lines* AG *and* DH, *making the angles* AGC *and* DHF *equal, or elſe unequal (yet neverthleſs given) which ſhall have to one another given ratio's* AG *to* DH, *thoſe triangles* ABC *and* DEF *ſhall have alſo a given ratio to one another, to wit,* ABC *to* DEF.

*Conſtr.* For let the parallelograms KC and LF be finiſhed.

*Demonſtr.* Foraſmuch as the angles AGC and DHF are equal, or unequal (yet given) and that the angle AGC (*a*) is equal to the angle KBC, and alſo the angle DHF equal to the angle LEF, the angles at the points B and E are equal, or elſe unequal (yet given,) and becauſe the ratio of AG to DH is given, and AG is equal to KB, and DH is equal to LE, therefore the ratio of KB to LE is given. But the ratio of BC to EF is alſo given, and the angles at the points B and E are equal, or elſe unequal (yet given:) Therefore (*b*) the ratio of the paralel-

a 29. 1.

b 70 *prop.*

parallelogram KC to the parallelogram LF is given; and therefore the ratio of the triangle ABC to the triangle DEF is given, ſeeing thoſe triangles (*c*) are the one half c 41. 1. of the parallelograms.

## PROP. LXXIII. Plate VIII. *Fig.* 38, 39.

*If of two parallelograms* AB *and* EH, *the ſides about the equal angles* C *and* F, *or elſe about the unequal angles (but nevertheleſs given) are in ſuch ſort to one another, that as the ſide* CB *of the firſt, is to the ſide* EH *of the ſecond; ſo the other ſide* EF *of the ſecond, is to ſome other right-line* CN. *But that the other ſide* AC, *hath alſo to the ſame right-line* CN *a given ratio, thoſe parallelograms will have alſo to one another a given ratio* AB *to* EH.

*Conſtr.* For in the firſt place, let the parallelogram AB be equiangled to EH, and having placed CN directly to AC: Let the parallelogram CM be finiſhed.

*Demonſt.* Foraſmuch then as CB or NM its equal, is to FH, ſo is EF to CN, and that the angles N and F are equal (for N is equal to the angle ACB, which is put equal to F) the parallelograms CM and EH (*a*) are a 14. 6. equal: But as AC to CN, ſo (*b*) the parallelogram AB is b 1. 6. to the parallelogram CM or EH: Therefore ſeeing that the ratio of AC to CN is given, the ratio of AB to EH is alſo given.

*Conſtr.* 2. Now ſuppoſe the parallelogram AB not to be equiangled to the parallelogram EH, and let there be conſtituted at the given point C in the line CB, the angle BCK, equal to the angle EFH, and ſo finiſh the parallelogram CL.

*Demonſtr.* 2. Seeing that each of the angles ACB and KCB is given, the remaining angle ACK is alſo given. But (*c*) the angle CAK is given, as alſo the remaining an- c *ſch.* 69. *prop.* gle AKC: Therefore (*d*) the triangle ACK is given in kind: and therefore the ratio of AC to CK is given. d 40. *prop.* But the ratio of the ſame AC to CN is alſo given: Therefore (*e*) the ratio of CK to CN is given. And ſee- e 8. *prop.* ing that as CB is to FG; ſo is EF to the right-line CN, to which the other ſide KC hath a given ratio, and that the angle BCK is equal to the angle F, the ratio of the parallelogram CL to the parallelogram EH is given (by the firſt part of this propoſition) but the parallelogram CL is equal to the parallelogram AB: Therefore the ratio of the parallelogram AB to the parallelogram EH is given.

## PROP. LXXIV. Plate VIII. *Fig.* 38, 39.

*If two parallelograms* AB *and* EH, *in equal angles* C *and* F, *or elſe in unequal angles (yet neverthelefs given angles) have a given ratio to one another, as one ſide* CB *of the firſt ſhall be to one ſide* FH *of the ſecond, ſo the other ſide* EF *of the ſecond, ſhall be to that to the which the other ſide* AC *of the firſt hath a given ratio.*

*Conſtr.* For either AB is equiangled or not; ſuppoſe it in the firſt place to be equiangled, and to the right-line BC let there be applied the parallelogram CM, equal to the parallelogram EH, and ſo poſited, as that AC and CN may be direct: Therefore (*a*) DB and BM ſhall be alſo direct (that is, as one right-line.) a *ſch.* 68. *prop.*

*Demonſtr.* Seeing that the ratio of AB to EH is given, and that CM is equal to EH, the ratio of AB to CM is alſo given; and therefore the ratio of AC to CN is given (ſeeing AB is to CM, (*b*) as AC is to CN;) and as CM is equal and equiangled to EH, the ſides about the equal angles of the parallelograms CM and EH. (*c*) are reciprocally proportional; and therefore as CB is to FH, ſo is EF to CN. But the ratio of AC to CN is given: Therefore as CB is to FH, ſo is EF to that to which AC hath a given ratio. b 1. 6. c 14. 6.

*Conſtr.* 2. Now ſuppoſe AB not to be equiangled to EH, and in the given point C of the line CB, let there be conſtituted the angle BCK equal to the angle EFH, and finiſh the parallelogram CL.

*Demonſtr.* 2. Seeing then that the ratio of AB to EH is given, and (*d*) that AB is equal to CL, alſo the ratio of CL to EH is given, and the angle BCK is equal to the angle F, and therefore CL (*e*) is equiangled to EH: Therefore (by the firſt part of this propoſition) as CB is to FH, ſo is EF to that to the which CK hath a given ratio. But the ratio of AC to CK is given; (as appears by what hath been demonſtrated in the latter part of the precedent propoſition.) Therefore as CB is to FH, ſo is EF to that to which AC hath a given ratio, d 36. 1. e *ſch.* 69. *prop.*

## PROP. LXXV. Plate VIII. *Fig.* 40, 41.

*If two triangles* ABC *and* DEF, *in equal angles* A *and* D, *or elſe unequal* (*yet neverthe leſs given*) *have to one another a given ratio, as the ſide* AB *of the firſt, ſhall be to the ſide* DE *of the ſecond, ſo the other ſide* DF *of the ſecond, ſhall be to that right-line to the which the other ſide* AC *of the firſt hath a given ratio.*

*Conſtr.* For let the parallelograms AG and DH be finiſhed.

*Demonſtr.* Foraſmuch as the ratio of the triangle ABC to the triangle DEF is given, alſo the ratio of the parallelogram AG to the parallelogram DH is given.

Seeing therefore that the two parallelograms AG and DH in equal angles, or unequal angles (nevertheleſs given) have to one another a given ratio; as (*a*) AB is to DE, ſo is DF to that to which AC hath a given ratio. a 74. *prop.*

## PROP. LXXVI. *Fig.* 4.

*If from the top* A *of a triangle* ABC, *given in kind, there be drawn to the baſe* BC, *a perpendicular line* AD, *that line* AD *ſhall have to the baſe* BC *a given ratio.*

*Demonſtr.* For ſeeing that the triangle ABC is given in kind, the ratio of AB to BC is given; and the angle B is alſo given. But the angle ADB is given; therefore the other angle BAD is given. Wherefore (*a*) the triangle ADB is given in kind: and therefore the ratio of A B to A D is given. But the ratio of AB to BC is given: Therefore (*b*) the ratio of AD to BC is given.

a 40. *prop.*
b 8. *prop.*

## PROP. LXXVII. *Fig.* 44, 45.

*If two figures* ABC *and* DEF, *given in kind, have to one another a given ratio, the ratio alſo ſhall be given of which you pleaſe of the ſides of one of the figures, to which you pleaſe of the ſides of the other figure.*

*Conſtr.* For on the right-lines BC and EF, let there be deſcribed the ſquares BG and EH.

*Demonſtr.* Foraſmuch as on one and the ſame right-line BC, are deſcribed two figures ABC and BG given in kind, (*a*) the ratio of the ſaid ABC to BG is given. In like manner the ratio of DEF to EH is given; and ſeeing that the ratio of ABC to DEF is given; and alſo that of the ſame figure ABC to BG; and again the ratio of DEF to EH: (*b*) the ratio of BG to EH is given; and therefore the ratio of BC to EF is alſo given.

a 49. *prop.*
b 8. *prop.*

PROP. LXXVIII. Plate VIII. *Fig.* 46, 47.

*If a given figure* ABC, *hath a given ratio to ſome rectangled figure* DF, *and that one ſide* BC *hath a given ratio to one ſide* DE, *the rectangled figure* DF *is given in kind.*

*Conſtr.* For on the right-line BC let the ſquare BH be deſcribed, and to the right-line DE, let the parallelogram DK be applied equal to BH, in ſuch a manner, as that GD and DI may be placed directly, (*a*) and by conſequence FE and EK alſo directly.

a *ſch.* 68.*prop.*

*Demonſtr.* Therefore ſeeing that on one and the ſame right-line BC are deſcribed the two rectiline figures ABC and BH, given in kind, (*b*) the ratio of ABC to BH is given. But the ratio of the ſaid ABC to DF is alſo given: Therefore (*c*) the ratio of BH to DF is given. But BH is equal to DK: Therefore the ratio of DK to DF is alſo given. And ſeeing that BH is equal and equiangled to DK, both the one and the other being rectangles, (*d*) the ſides of thoſe figures are reciprocally porportional; and as BC is to DE, ſo is DI to CH. But by ſuppoſition, the ratio of BC to DE is given; therefore alſo the ratio of DI to CH is given; but the ratio of DI to DG is alſo given: (for DI is to DG (*e*) as DK to DF:) Therefore (*f*) the ratio of DG to CH is given. But CH is equal to BC, ſeeing that BH is a ſquare; therefore the ratio of the ſame BC to DG is given: But the ratio of the ſame BC to DE is alſo given; therefore the ratio of DE to DG is given, and the angle at D is a right angle: Therefore (*g*) DF is given in kind,

b 49, *prop.*
c 8. *prop.*
d 14. 6.
e 1. 6,
f 8. *prop.*
g *ſch.* 61.*prop*

PROP.

## PROP. LXXIX. Plate VIII. *Fig.* 48, 49.

*If two triangles* ABC *and* EFG, *have an angle* B *equal to an angle* F. *And from the equal angles* B *and* F *there be drawn perpendiculars* BD *and* FH, *to the baſes* AC *and* EG; *and that as the baſe* AC *of the firſt triangle* ABC, *is to the perpendicular* BD, *ſo alſo the baſe* EG *of the other triangle* EFG, *is to the perpendicular* FH, *thoſe triangles* ABC *and* EFG *are equiangled.*

*Conſtr.* For about the triangle EFG let there be deſcribed the circle EFLG, then on the right-line EG, and in the point E given therein, let there be made the angle GEL, equal to the angle C, and let FL and LG be drawn, and the perpendicular LM.

*Demonſtr.* Seeing then that the angle GEL is equal
to the angle C, and the angle ELG is equal to the angle
EFG, (*a*) they being in one and the ſame ſegment of the a 21. 3.
circle; the third angle EGL is equal to the third angle
A. Wherefore the triangle ABC is alike to the triangle
ELG, and the perpendiculars BD and LM are drawn:
Therefore † as AC is to BD, ſo is EG to LM; but by
ſuppoſition as AC is to BD; ſo is EG to FH: Therefore
(*b*) LM is equal to FH. But the ſaid LM is (*c*) parallel b 7. 5.
to FH: Therefore (*d*) FL is alſo parallel to EG; and c 28. 1.
therefore the angle FLE (*e*) is equal to the angle LEG. d 33. 1.
But the angle C is alſo equal to the ſaid angle LEG, and e 29. 1.
the angle FLE to the angle FGE (*f*): Therefore alſo the f 21. 3.
angle C is equal to the angle FGE. But by ſuppoſition
the angle ABC is equal to the angle EFG: Therefore
the third angle BAC is equal to the third angle FEG:
Wherefore the triangle ABC is equiangled to the triangle EFG.

### Scholium.

† *Now that as* AC *is to* BD, *ſo* EG *is to* LM, *it is by ſome thus demonſtrated. Foraſmuch as the angle* C *is equal to the angle* GEL, *and the angle* BDC *to the angle* LME, *each being a right-angle, the other angle* CBD *is equal to the other angle* ELM: *Therefore* (g) *as* EM *is to* g 4. 6.
ML, *ſo is* CD *to* DB. *Again, ſeeing the angle* ABC *is equal to the angle* ELG, *and the angle* CBD *to the angle* ELM, *the remaining angle* ABD *is equal to the remaining angle* MLG; *but the angle* ADB *is alſo equal to the angle* LMG; *and therefore the third angle* A *is equal to the*

h 4. 6. *third angle* LGM : *Therefore* (h) *as* AD *is to* DB, *ſo is*
GM *to* ML. *But it hath been demonſtrated, that as* CD
i 14. 5. *is to* DB, *ſo is* EM *to* ML : *Therefore* (i) *as* AC *is to* BD,
*ſo is* EG *to* LM.

## PROP. LXXX. Plate VIII. *Fig.* 50, 51.

*If a triangle* ABC *hath one angle* A *given, and that the rectangle contained under the ſides* AB *and* AC, *compriſing the given angle* A, *hath a given ratio to the ſquare of the other ſide* BC, *the triangle* ABC *is given in kind.*

*Conſtr.* For from the points A and B, let there be drawn the perpendiculars AD and BE.

*Demonſtr.* Foraſmuch as the angle BAE is given, and
a 40. *prop.* alſo the angle AEB, the triangle ABE is given in (*a*)
kind ; and therefore the ratio of AB to BE is given :
Therefore the ratio of the rectangle of AB and AC to
the rectangle of BE and AC is alſo given (for it is the
b 1. 6. ſame ratio (*b*) as of AB to BE.) But the rectangle of
AC and BE is equal to the rectangle of BC and AD ;
c 41. 1. for that each of thoſe rectangles is (*c*) double to the
triangle ABC. Therefore the ratio of the rectangle of AB
and AC to the rectangle of BC and AD is alſo given.
But the ratio of the rectangle of AB and AC to the
d 8. *prop.* ſquare of BC is given : Therefore (*d*) alſo the ratio of
the rectangle of BC and AD to the ſquare of BC is
given ; and therefore the ratio of the right-line BC to
e 1. 6. the right-line AD is given. (For that (*e*) the rectangle
is to the ſquare as AD to BC.) Now let the right-line
FD, given in poſition and magnitude, be expoſed ;
and thereon let there be deſcribed the ſegment of a
circle FID ; capable of an angle equal to the angle A.
And ſeeing the ſaid angle A is given, alſo the angle in
f 8. *def.* the ſegment FLD ſhall be given ; and therefore (*f*) the
ſame ſegment is given in poſition. From the point D
let there be erected at right-angles on the line FD, the
g 4. *def.* line DH, which (*g*) is given in poſition : Let it be ſo
made, that as BC is to AD, ſo FD may be to DH and
ſeeing that the ratio of BC to AD is given, alſo that of
h 2. *prop.* FD to DH is given But FD is given : Therefore (*h*)
DH is given in magnitude. But it is alſo given in poſition,
and the point D is given : Therefore the point
i 27. *prop.* H is (*i*) alſo given. Now through the point H let
there be drawn HI, parallel to FD, and that line HI
k 28. *prop.* ſhall be given in (*k*) poſition. But the ſegment of the
l 25. *prop.* circle FID is alſo given in poſition. Therefore (*l*) the
point

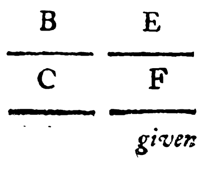
point I is given. Let the right-lines IF and ID be
drawn, and the perpendicular IE : Therefore IE is given
in poſition. But the point I is given, as alſo each of
the points F and D: Therefore (*m*) each of the lines m 26. *prop.*
FD, FI, and ID is given in poſition and magnitude :
Wherefore (*n*) the triangle FID is given in kind; and n 39. *prop.*
ſeeing that as BC is to AE, ſo is FD to DH, and (*o*) o 34. *prop.*
that to DH, IE is equal; as BC is to AE, ſo is FD
to IE, and the angle A is equal to the angle FID :
Therefore (*p*) the triangle ABC is equiangled to the p 79. *prop.*
triangle FID. But FID is given in kind : Therefore
alſo the triangle ABC is given in kind.

## OTHERWISE. *Fig.* 52, 53.

*Conſtr.* Let the triangle ABC, whoſe angle A is given, and the ratio of the rectangle contained under AB and AC, to the ſquare of BC be given : I ſay that the triangle ABC is given in kind.

*Demonſtr.* For ſeeing the angle A is given, that ſpace
by which the ſquare of the line compounded of BAC is
greater than the ſquare of BC, (*q*) hath a given ratio q 67. *prop.*
to the triangle ABC. Now let that ſpace be D : Therefore the ratio of D to the triangle ABC is given. But
the ratio of the triangle ABC to the rectangle of AB and
AC is given ; (*r*) ſeeing the angle A is given : There- r 66. *prop.*
fore (*s*) the ratio of the ſpace D to the rectangle of s 8. *prop.*
AB and AC is given. But the ratio of the rectangle
of AB and AC to the ſquare of BC is alſo given :
Therefore (*s*) the ratio of the ſpace D to the ſquare of
BC is given. Wherefore by compounding, (*t*) the ratio t 6. *prop.*
of the ſpace D, with the ſquare of BC the ſaid ſquare
of BC is given : Therefore the ratio of the ſquare of the
line compounded of BAC, to the ſquare of BC is given;
(for that the ſpace D with the ſquare of BC is equal u *ſch.* 52.
to the ſquare of the line compounded of BAC;) and *prop.*
therefore (*u*) the ratio of the ſaid line compounded of x 46. *prop.*
BAC to BC is given. But the angle A is alſo given :
Therefore (*x*) the triangle ABC is given in kind.

## PROP. LXXXI.



*If of three right-lines* A, B, *and* C, *proportional to three other proportional right-lines* D, E, *and* F, *the extremes* A *and* D, C *and* F, *are in a given ratio,* (*to wit, as* A *to* D, *and* C *to* F,) *alſo the means,* B *and* E *ſhall be in a given ratio, and if one extreme hath a given*

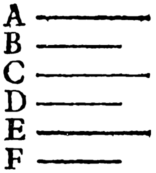
*given ratio to an extreme, and the mean to the mean, the other will have alſo a given ratio to the other.*

*Demonſtr.* Foraſmuch as the ratio of A to D, and of
a 70. *prop.* C to F is given, the rectangle of A and D (*a*) ſhall have
a given ratio to the rectangle of C and F. But the
b 17. 6. rectangle of A and D is equal (*b*) to the ſquare of B; and
the rectangle of C and F to the ſquare of E. Therefore
the ratio of the ſquare of B to the ſquare of E is given;
c *ſch.* 52. *prop.* and therefore (*c*) the ratio of the line B to the line E is
alſo given.

Again, let the ratio of A to D, and B to E, be given: I ſay that the ratio of C to F is alſo given. For ſeeing that the ratio of A to D, and of B to E is given, alſo
d 50. *prop.* the ratio of the ſquare of B (*d*) to the ſquare of E is given. But the ſquare of B is equal to the rectangle of A and C, and the ſquare of E to the rectangle of D and F: Therefore the ratio of the rectangle of A and C to the rectangle of D and F is given. But the ratio of a ſide
e 68. *prop.* A to a ſide D is given: Therefore (*e*) the ratio of the other ſide C to the other ſide F is alſo given.

## PROP. LXXXII.

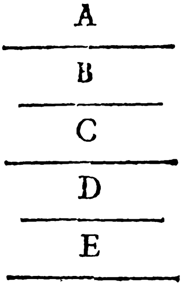
*If there be four right-lines* A, B, C, *and* D, *proportional, as the firſt* A *ſhall be to that line to which the ſecond* B *hath a given ratio, ſo the third* C *ſhall be to that to which the fourth* D *hath a given ratio.*



*Conſtr.* Let E be the line to which B hath a given ratio, and let it be ſo as that B may be to E, as D is to F.

*Demonſtr.* Now the ratio of B to E is given, therefore alſo the ratio of D to F is given. And ſeeing that as A is to B, ſo is C to D. And again, as B is to E, ſo is D to F, by ratio of equality, as A is to E, ſo is C to F. But E is that line to which B hath a given ratio, and F that to which D alſo hath a given ratio: Therefore as A is to that to which B hath a given ratio, ſo C is to that to which D hath a given ratio.

## PROP. LXXXIII.



*If four right-lines* A, B, C, *and* D, *are in ſuch ſort to one another, that of any three of them* A, B, C, *and a fourth* E, *taken proportional, to which that line* D, *which remains of the four lines, hath a given ratio, it ſhall be as the fourth* D *is to the third* C, *ſo the ſecond* B *ſhall be to that to which the firſt* A *hath a given ratio.*

*Demonſtr.* Foraſmuch as A is to B, as C is to E, the rectangle contained under A and E (*a*) is equal to the a 16. 6.
rectangle contained under B and C; and ſeeing that the ratio of D to E is given, alſo ſhall be given the ratio of the rectangle of A and D to the rectangle of A and E (for (*b*) it is the ſame ratio as of D to E.) But the rectangle of A and E is equal to the rectangle of B and C. b 1. 6.
Therefore the ratio of the rectangle of A and D to the rectangle of B and C is given. Wherefoie (*c*) as D is to C, ſo is B to that to which A hath a given ratio. c 56. *prop.*

## PROP. LXXXIV. Plate VIII. *Fig.* 54.

*If two right-lines* AB *and* AE *comprehending a given ſpace* AF *in a given angle* BAE, *and that the one* AB *be greater than the other* AE *by a given line* CB, *alſo each of the lines* AB *and* AE *is given.*

*Demonſtr.* For ſeeing that AB is greater than AE by the given line CB, the remainder AC is equal to AE: Finiſh the parallelogram AD. Therefore ſeeing that AE is equal to AC, the ratio of AE to AC is given, and the angle A is alſo given: Therefore (*a*) AD is given a *ſch.* 61. *prop.*
in kind. Wherefore the given ſpace AF is applied to the given right-line CB, exceeding it by the given figure AD given in kind; and therefore (*b*) the breadth b 59. *prop.*
of the exceſs is given. Therefore AC is given. But CB is alſo given: Therefore the whole AB is given. But AE is alſo given: Therefore each of the right-lines AB and AE is given.

## PROP. LXXXV. Plate VIII. *Fig.* 54.

*If two right-lines* AC *and* CD, *comprehend a given ſpace* AD *in a given angle* ACD, *the line compounded of thoſe lines* AC *and* CD *is given, alſo each of thoſe lines* AC *and* CD *is given.*

*Conſtr.* For let AC be prolonged to the point B, and let CB be put equal to CD, then through the point B let BF be drawn parallel to CD, and ſo finiſh the parallelogram CF.

*Demonſtr.* Seeing then that CB is equal to CD, and
the angle DCB is given ; for that angle that follows
a *ſch.* 61. *prop.* is the given angle ; and therefore (*a*) the parallelogram
DB is given in kind : and again, ſeeing that the line
compounded of ACD is given, and CB is equal to
CD, alſo AB is given. And thus to the right-line AB
there is applied the given ſpace AD, deficient by the
b 58. *prop.* figure DB given in kind : and therefore (*b*) the breadths
of the defects are alſo given : Therefore the right-lines
DC and CB are given. But the compounded line
c 4. *prop.* ACD is alſo given : Therefore (*c*) each of the lines AC
and CD is given.

## PROP. LXXXVI. *Fig.* 55.

*If two right-lines* AB *and* BC, *comprehend a given ſpace* AC, *in a given angle* ABC, *the ſquare of the one* BC, *is greater than the ſquare of the other* AB, *by a given ſpace (yet in a given ratio,) alſo each of thoſe lines* AB *and* BC *ſhall be given.*

*Demonſtr.* For ſeeing that the ſquare of BC is greater
than the ſquare of AB by a given ſpace (yet in a certain ratio:) Let the given ſpace be taken away, that
is to ſay, the rectangle contained under CB and BE :
a 11. *def.* Therefore (*a*) the ratio of the remainder, (*b*) which is the
b 2. 2. rectangle contained under BC and CE to the ſquare
of AB is given. And foraſmuch as the rectangle
† under AB and BC is given, and alſo that of CB and
c 1. *prop.* BE, their (*c*) ratio is given. But as the rectangle under
d 1. 6. AB and BG is to the rectangle under CB and EB, (*d*) ſo
AB is to BE ; and therefore the ratio of AB to BE is
e 50. *prop.* given : Wherefore (*e*) the ratio of the ſquare of AB to
the ſquare of BE is alſo given. But the ratio of the
ſquare of AB to the rectangle under BC and CE is given : Therefore (*f*) alſo the ratio of the rectangle under
f 8. *prop.*

BC and CE to the ſquare of BE is given. Wherefore
the ratio of four times the rectangle under BC and CE
to the ſquare of BE is given; and by compounding, (*g*) g 6. *prop.*
the ratio of four times the rectangle under BC and CE,
with the ſquare of BE the ſquare of BE is given
But four times the rectangle of BC and CE, with the
ſquare of BE, (*h*) is the ſquare of the compound line h 8. 2.
BCE: Therefore the ratio of the ſquare of the compound
line BCE to the ſquare of BE is given: Wherefore (*i*) the i 54. *prop.*
ratio of the line compounded of BC and CE to BE is
given, and by compounding (*k*) the ratio of the compound k 6. *prop.*
of the lines BC, CE, and BE, that is to ſay,
the double of BC to BE is given; and therefore the
ratio of the only line BC to BE is alſo given. But as
BC is to BE, (*l*) ſo the rectangle under BC and BE is l 1. 6,
to the ſquare of BE: Therefore the ratio of the rectangle
under BC and BE to the ſquare of BE is given.
But the rectangle of BC and BE is given: Therefore (*m*) m 2. *prop.*
the ſquare of BE is alſo given, and conſequently the
line BE is given. Wherefore BC is alſo given, ſeeing
that the ratio of BE to BC is given. But the ſpace
AC is given, and alſo the angle B: Therefore (*n*) AB n 57. *prop.*
is given. Wherefore each of the lines AB and BC is
given.

### Scholium.

† *Inſtead of ſaying in this place* [what is under, *&c.*] *we have uſed this word rectangle, it being manifeſt by what follows that ſuch was the intention of EUCLIDE, ſeeing that he makes uſe in the ſaid Demonſtration of the ſecond, and eighth propoſition of the ſecond Element; and alſo that the ſpace or parallelogram being given not rectangled, it may be reduced thereto, making on* BC, *and in the given point* B, *a right-angle* CBA, *ſo as that there will be two parallelograms conſtitued on one and the ſame baſe* BC, *and between the ſame parallels, as in the* 69*th propoſition, by means whereof this concluſion is drawn.*

*Note,* This ſerves alſo for the next Prop.

### PROP. LXXXVII. *Fig.* 56.

*If two right-lines* AB *and* BC, *comprehend a given ſpace* AC, *in a given angle* B, *the ſquare of the one* BC *is greater than the ſquare of the other* AB, *by a given ſpace; alſo each of thoſe lines* AB *and* BC *ſhall be given.*

*Demonſtr.* For ſeeing that the ſquare of BC is greater
than the ſquare of AB by a given ſpace: Let the given
ſpace be taken away, and let the rectangle be contained
a 2. 2. under BC and BE: Therefore the remainder (*a*) which
is the rectangle of BC and CE, is equal to the ſquare of
AB. And ſeeing that the rectangle of BC and BE is
given, and alſo the ſpace or rectangle AC, the ratio of
the ſaid rectangle of BC and BE to AC is given. But
b 1. 6. as (*b*) the rectangle of BC and BE is to the rectangle
of AB and BC, ſo is BE to AB: Therefore the ratio of
c 50. *prop.* BE to AB is given, and therefore (*c*) the ratio of the
ſquare of the ſaid DE to the ſquare of AB is alſo given.
But to that ſquare of AB the rectangle of BC and CE
is equal: Therefore the ratio of the ſaid rectangle of
BC and CE to the ſquare of BE is given; and therefore
the ratio of the quadruple of the ſaid rectangle of BC
and CE to the ſquare of BE is alſo given; and by
d 6. *prop.* compounding (*d*) the ratio of four times the rectangle
of BC and CE, with the ſquare of BE to the ſaid
ſquare of BE is given. But four times the rectangle of
e 8. 2. BC and CE, with the ſquare of BE, (*e*) is the ſquare of
the compound line BCE: Therefore the ratio of the
ſquare of that compound line BCE to the ſquare of BE is
f 54. *prop.* alſo given; and therefore the ratio (*f*) of the compound
line BCE to BE is given. Wherefore by compounding
g 6. *prop.* (*g*) the ratio of the ſaid compound line BCE and EB,
that is to ſay, twice BC to BE is alſo given; therefore
the ratio of the only line BC to BE is given. But the
ratio of the ſame BE to AB is alſo given: Therefore
h 8. *prop.* (*h*) the ratio of AB to BC is given. And ſeeing that the
ratio of BC to BE is given, and that as the ſaid BC is
i 1. 6. to BE, ſo is the ſquare of BC (*i*) to the rectangle of BC
and BE, the ratio of the ſquare of BC to the rectangle
of BC and BE is alſo given. But the ſaid rectangle of
BC and BE is given, it being that which was taken
away, and which was given. Therefore the ſquare of
k 2. *prop.* BC (*k*) is given, and therefore the line BC is given.
But the ratio of the ſame BC to BA is given, therefore
AB is alſo given.

## PROP. LXXXVIII. Plate VIII. *Fig.* 57.

*If in a circle* ABC, *given in magnitude, there be drawn a right-line* AC, *which ſhall take away a ſegment* ABC, *which doth comprehend a given angle* AEC, *that line* AC *is given in magnitude.*

*Conſtr.* For let D be the center of the circle; and let the diameter thereof ADE be drawn, and let EC be joined.

*Demonſtr.* The angle ACE is given for (*a*) it is a right-angle. But the angle AEC is alſo given, and therefore the other angle CAE is given. Wherefore the triangle ACE (*b*) is given in kind; and therefore the ratio of EA to AC is given. But AE is given in magnitude, ſeeing that the circle ABC is given in magnitude. Therefore (*c*) AC is alſo given in magnitude.

a 31. 3.
b 40. *prop.*
c 2. *prop.*

## PROP. LXXXIX. Plate VIII. *Fig* 57.

*If in a circle* ABC, *given in magnitude, there be drawn a right-line* AC, *given in magnitude, that line* AC *will take away a ſegment* ABC, *comprehending a given angle.*

*Conſtr.* For having taken the point D for the center of the circle, let the diameter ADE be drawn, as alſo the right-line EC.

*Demonſtr.* Foraſmuch as each of the right-lines AE and AC are given, the ratio of the line AE to AC (*a*) is given; and the angle ACE is a right-angle: Therefore (*b*) the triangle ACE is given in kind, and therefore the angle AEC is given.

a 1. *prop.*
b 43. *prop.*

## PROP. XC. *Fig.* 58.

*If in the circumference of a circle* ABC, *given in poſition and in magnitude, there be taken a given point* B, *and that from the point* B *to the circumference of the circle* ABC, *a right-line* BAC *be inflected ſo as to make a given angle* BAC, *the other extremity* C *of the inflected-line ſhall be given.*

*Conſtr.* For let the center of the circle be D, and let the right-lines BD and BC be drawn.

*Demonſtr.* Foraſmuch as each point B and D is given, the right-line BD, (*a*) is given in poſition; and ſeeing that the angle BAC is given, the angle BDC is alſo given. Wherefore to the right-line BD, given in poſition, and in the point D given therein, there is drawn the right-line CD; which makes the given angle BDC; and therefore (*b*) the line DC is given in poſition. But the circle ABC is given in poſition and magnitude: Therefore (*c*) the right-line DC is given in poſition and in magnitude. But the point is D given: Therefore (*d*) the point C is alſo given.

a 26. *prop.*
b 29. *prop.*
c 6. *def.*
d 27. *prop.*

PROP.

## PROP. XCI. Plate VIII. *Fig.* 59.

*If from a given point* C, *there be drawn a right-line* CA, *which ſhall touch a circle* AB, *given in poſition; that line* CA *is given in poſition and in magnitude.*

*Conſtr.* For having taken the point D for the center of the circle, let the right-lines DA and DC be drawn.

a 26. *prop.* b 18. 3. *Demonſtr.* Foraſmuch as each point C and D is given, the right-line CD (*a*) is given in poſition and in magnitude. But the angle CAD (*b*) is a right-angle; and therefore the ſemicircle deſcribed on CD ſhall paſs through the point A: Let it then paſs through that point, and let the ſemicircle be DAC: Foraſmuch as c 6 *def.* the ſame DAC (*c*) is given in poſition, and alſo the d 25. *prop.* circle ABE, (*d*) the point A is given. But the point e 26. *prop.* C is alſo given: Therefore (*e*) the right-line AC is given in poſition and in magnitude.

## PROP. XCII. *Fig.* 60.

*If without a circle* ABC, *given in poſition, there be taken ſome point* D, *and from that given point there be drawn a right-line* DB, *cutting the circle, the rectangle compriſed under the whole line* BD, *and the part* DC, *between the point* D, *and the convexity of the circumference* AC, *ſhall be given.*

*Conſtr.* For from the point D let the right-line DA be dtawn, which ſhall touch the circle in the point A.

a 91. *prop.* *Demonſtr.* Therefore DA (*a*) is given in poſition and magnitude; and therefore the ſquare of the ſaid DA is b 52. *prop.* (*b*) given. But the ſaid ſquare of DA is equal (*c*) to the c 36. 3. rectangle of BD and DC: Therefore the ſaid rectangle of BD and DC is alſo given.

## OTHERWISE. *Fig.* 61.

*Conſtr.* Let E be the center of the circle, and through the ſame center let there be drawn from the point D the right-line DA.

*Demonſtr.* Foraſmuch as each point D and E is given, d 26. *prop.* the right-line DE is (*d*) given in poſition and in magnitude. But the circle ABC is given in poſition and in e 25. *prop.* magnitude: Therefore each point A and F (*e*) is given, and the point D is alſo given; and therefore each line

Plate VIII. *Facing Pag. 320.*

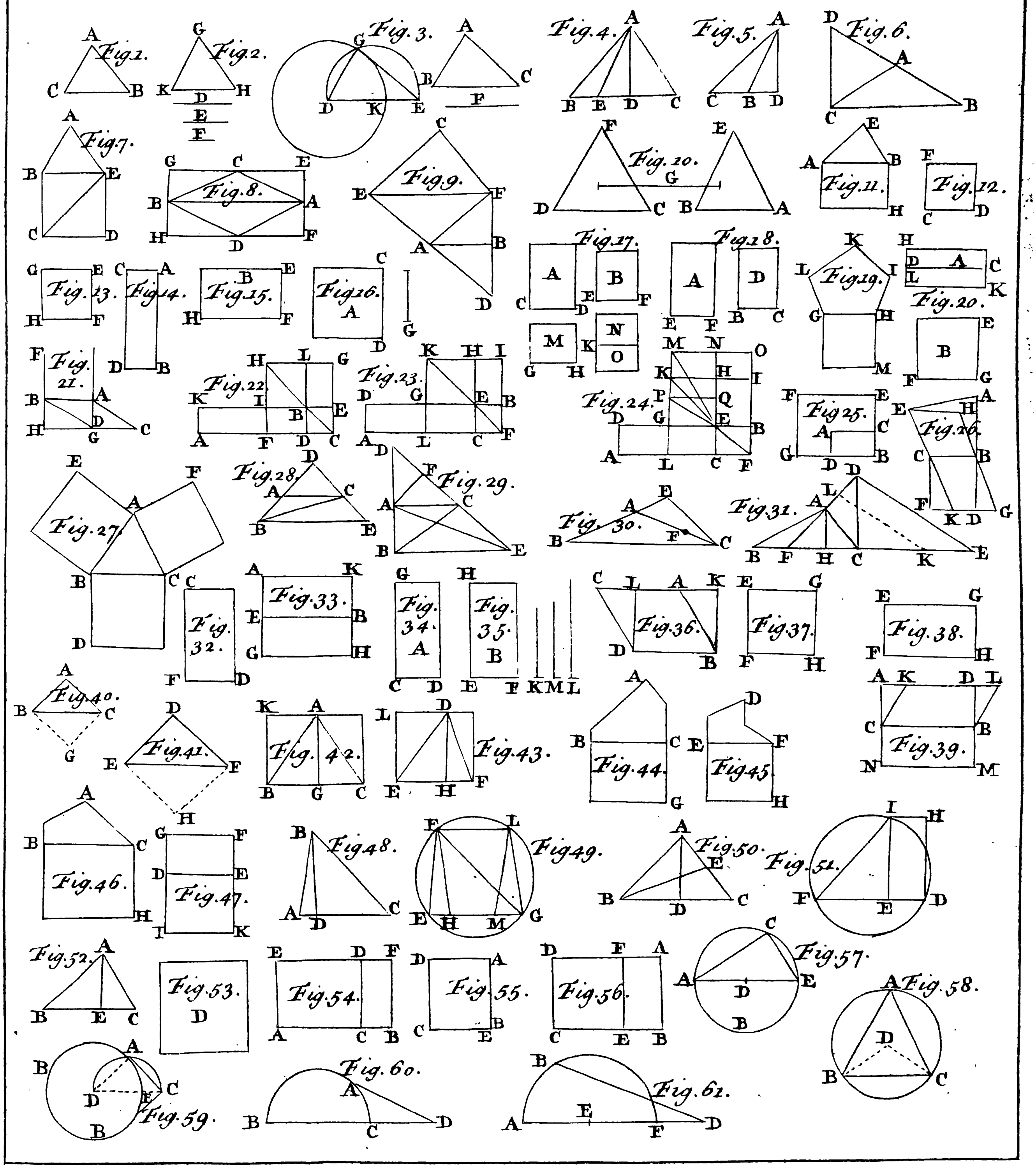


Fig. 1. A C B
Fig. 2. G K H D E F
Fig. 3. G D K E B A C F
Fig. 4. A B E D C
Fig. 5. A C B D
Fig. 6. D A C B
Fig. 7. A B E C D
Fig. 8. G C E B A H D F
Fig. 9. C E F A B D
Fig. 10. F D C G E B A
Fig. 11. E A B H
Fig. 12. F C D
Fig. 13. G E H F
Fig. 14. C A D B
Fig. 15. B E H F
Fig. 16. C A D G
Fig. 17. A B C D E F M N O K G H
Fig. 18. A D E F B C
Fig. 19. K L I G H M
Fig. 20. H D A C L K B E F G
Fig. 21. F B A H G D C
Fig. 22. H L G K I B E A F D C
Fig. 23. K H I D G E B A L C F
Fig. 24. M N O K H I P Q D G E B A L C F
Fig. 25. F E A C G D B
Fig. 26. E H A C B F K D G
Fig. 27. E A F B C D
Fig. 28. D A C B E
Fig. 29. D F A C B E
Fig. 30. A E B F C
Fig. 31. L D A B F H C K E
Fig. 32. C F D
Fig. 33. A K E B G H
Fig. 34. G A C D
Fig. 35. H B E F K M L
Fig. 36. C L A K D B
Fig. 37. E G F H
Fig. 38. E G F H
Fig. 39. A K D L C B N M
Fig. 40. A B C G
Fig. 41. D E F H
Fig. 42. K A B G C
Fig. 43. L D E H F
Fig. 44. A B C G
Fig. 45. D E F H
Fig. 46. A B C H
Fig. 47. G F D E I K
Fig. 48. B A D C
Fig. 49. F L E H M G
Fig. 50. A E B D C
Fig. 51. I H F E D
Fig. 52. A B E C
Fig. 53. D
Fig. 54. E D F A C B
Fig. 55. D A C E B
Fig. 56. D F A C E B
Fig. 57. C A D E B
Fig. 58. A D B C
Fig. 59. A B D E C B
Fig. 60. A B C D
Fig. 61. B E A F D

AD and FD is given. Wherefore the rectangle of the lines AD and DF is alſo given. But the ſaid rectangle of AD and DF is equal to the rectangle of DB and DC: Therefore the rectangle of DB and DC is given.

### PROP. XCIII. Plate IX. *Fig.* 1.

*If in a circle given in poſition there be taken a given point* A, *and through that point* A *there be drawn a right-line* BC *to [the circle, the rectangle compriſed under the ſegments the ſame line* BC, *ſhall be given.*

*Conſtr.* For let D be taken for the center of the circle, and having drawn the right-line AD prolong it to the point E and F.

*Demonſtr.* Foraſmuch as each point A and D is given, the right-line AD (*a*) is given in poſition. But the circle a 26. *prop.* BEC is alſo given in poſition: Therefore each point E and F is alſo given in poſition, and the point A is given. Wherefore each line (*b*) AE and AF is given: b 35. 3. Therefore the rectangle of the ſame lines AE and AF is given, and is equal to the rectangle (*b*) of AB and AC: Therefore the ſaid rectangle of AB and AC is given.

### PROP. XCIV. *Fig.* 2.

*If in a circle* ABC, *given in magnitude, there be drawn a right-line* BC, *which takes away a ſegment which comprehends a given angle* BAC, *and that the ſaid angle in the ſegment is cut into two equal parts, the line compounded of the right-lines* BA *and* AC, *which comprehend the given angle* BAC *ſhall have a given ratio to the line* AD, *which divides that angle into two equal parts; and the rectangle contained under the line compounded of thoſe lines* BA *and* AC, *comprehending the given angle* BAC, *and that part* ED *of the interſecting-line which is below the ſegment between the baſe* BC *and the circumference, ſhall be given.*

*Conſtr.* Let BD be drawn.

*Demonſtr.* Foraſmuch as in the circle ABC given in magnitude, there is drawn the right-line BC, which takes away the ſegment BAC, comprehending the given angle BAC, that line BC (*a*) is given; and therefore BD a 88. *prop.* is alſo given: Therefore the ratio of BC to BD (*b*) is b 1. *prop.* given. And ſeeing that the given angle BAC is cut in two equal parts by the right-line AD, as (*c*) BA c 3. 6. is to CA, ſo is BE to CE; and by compounding, as

BAC is to CA, ſo is BC to CE; and by permutation,
as BAC is to BC, ſo is CA to CE. And ſeeing that
the angle BAE is equal to the angle CAE, and the
d 21. 3. angle ACE (*d*) to the angle BDE, the other angle AEC
is equal to the other angle ABD; and therefore the
triangle ACE is equiangled to the triangle ABD:
e 4. 6. Therefore (*e*) as AC is to CE, ſo is AD to BD. But
as CA is to CE, ſo the line compounded of BA and
AC is to BC. Therefore as the compound-line BAC
is to BC, ſo is AD to BD; and by permutation, as the
compound-line BAC is to AD, ſo is BC to BD. But
the ratio of BC to BD is given: Therefore the ratio of
the compound-line BAC to AD is alſo given. Moreover,
I ſay that the rectangle under the compound-line
BAC and ED is given. For ſeeing that the triangle
AEC is equiangled to the triangle BDE, (for the angle
ACE (*d*) is equal to the angle BDE, and the angle AEC
f 15. 1. (*f*) to the angle BED) as BD is to DE, ſo is AC to CE.
But as AC is to CE, ſo is alſo the compound-line BAC
to BC: Therefore as the compound-line BAC is to BC,
ſo is BD to DE. Wherefore the rectangle of the compound-line
g 16. 6. BAC and DE (*g*) is equal to the rectangle
of BC and BD. But the rectangle of BC and BD is
given, (for that thoſe lines BC and BD are given:)
Therefore the rectangle under the compound-line BAC
and ED is alſo given.

### OTHERWISE. Plate IX. *Fig.* 4.

*Conſtr.* Let CA be prolonged to the point E, and let AE be put equal to BA, and let BE and BD be joined.

*Demonſtr.* Foraſmuch as the angle BAC is double to
each of the angles CAD and AEB (for the angle BAC
is cut into two equal parts by the line AD, and equal
h 32. 1. (*h*) to the two angles ABE and AEB, which (*i*) are
i 5. 1. equal) the angle ABE is equal to the angle CAD, that
k 21. 3. is to ſay, (*k*) to the angle CBD; adding therefore the
common angle ABC, the whole angle ABD ſhall be
equal to the whole angle FBE. But the angle ACB
is (*k*) equal to the angle ABD: Therefore the third
angle AEB is equal to the third angle BAD; and
therefore the triangle CEB is equiangled to the triangle
ABD: Wherefore as CE is to CB, ſo is AD to
BD. But the right-line CE is compounded of the two
lines CA and AB: Therefore as the compound-line
BAC is to CB, ſo is AD to BD; and by permutation,

as the compound-line BAC is to AD, ſo is CB to BD.
But the ratio of CB to BD is given, ſeeing that each of
thoſe lines is given: Therefore the ratio of the compound-line
BAC to AD is alſo given. And ſeeing that
the triangle CEB is equiangled to the triangle FBD
(for the angle AFC is equal (*l*) to the angle BFD, l 21. 3.
the angle ECB (*m*) to the angle ADB) as EC is to m 16. 6.
CB, ſo is BD to DF. But EC is equal to the compound-line
BAC: Therefore as the compound-line
BAC is to CB, ſo is BD to DF. Wherefore (*n*) n 16. 6.
the rectangle of the compound-line BAC and DF is
equal to the rectangle of CB and BD. But the rectangle
of CB and DC is given, ſince each of the lines
CB and BD is given; Therefore the rectangle of the
compound-line BAC and DF is given.

## OTHERWISE. Plate IX. *Fig.* 5.

*Conſtr.* Let AC be prolonged to F, and let CF be put equal to AB, and let the right-lines BD and DF be drawn.

*Demonſtr.* Foraſmuch as BA is equal to CF, and (*o*) BD o 26, 29. 3.
to DC, the two ſides AB and BD are equal to the two
ſides CD and DF, each to his correſponding ſide, and the
angle ABD is equal to the angle DCF, (*p*) ſeeing that the p 22. 3.
four ſided figure ABDC is within the circle: Therefore
the baſe AD is (*q*) equal to the baſe DF, and the angle q 4. 1.
DAB to the angle DFC. But the angle BAD is given,
being the half of the given angle BAC, therefore the
angle DFC is ſo alſo. But DAF is alſo given: Therefore
the triangle ADF is given in kind. Wherefore
the ratio of FA to AD is given. But AF is the compound
of BA and AC, for that CF is equal to AB:
Therefore the ratio of the compound-line BAC to AD
is given: The ſame demonſtration will ſerve to ſhew
that the rectangle contained under the compound-line
BAC and ED is given alſo.

## PROP. XCV. *Fig.* 6.

*If in the diameter* BC *of a circle* ABC *given in poſition, there be taken a given point* D, *and from that point* D *there be drawn a right-line* DA, *to the circumference of the circle. And if from the ſection of the ſaid line there be drawn a right-line* AE, *perpendicular thereto, and through the point* E *where that perpendicular meets with* *the circumference, there be drawn a parallel* EF, *to the firſt line drawn* AD, *that point* F *in which the parallel meets with the diameter, is given; and the rectangle contained under the parallel-lines* AD *and* EF *is alſo given.*

*Conſtr.* Let the right-line EF be prolonged to the point G, and let the right-line AG be drawn.

*Demonſtr.* Foraſmuch as the angle AEG is a right-angle, the right-line AG is the diameter of the circle. But BC is alſo the diameter: Therefore the point H is the center of the circle. Now the point D is given;
a 26. *prop.* and therefore (*a*) the line DH is given in magnitude. But ſeeing that AD is parallel to EG, and AH equal to GH;
b 26. *prop.* (*b*) DH is equal to FH, and AD to FG: (for the angles
c 15. 1. AHD and FHG (*c*) are equal, and DAH and FGH (*d* are
d 26. 1. alſo equal.) But the line DH is given: Therefore FH is alſo given. But each of thoſe lines DH and HF is alſo given in poſition, and the point H is given: Therefore
e 27. *prop.* (*e*) the point F is alſo given. And ſeeing that in the circle ABC given in poſition, is taken the given point F, and through the ſame is drawn the right-line
f 93. *prop.* EFG; the rectangle under EF and FG (*f*) is given. But FG is equal to AD. Therefore the rectangle comprehended under AD and EF is given. *Which was to be demonſtrated.*

## *The End of* EUCLIDE'S *DATA.*

# A
# BRIEF TREATISE
## (Added by *FLUSSAS*)
## OF
# Regular Solids.

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REgular Solids are ſaid to be compoſed and mix'd when each of them is transformed into other Solids, keeping ſtill the form, number and inclination of the baſes, which they before had to one another; ſome of which yet are transformed into mix'd Solids, and others into ſimple. Into mix'd, as a Dodecaedron and an Icoſaedron, which are transformed or altered, if you divide their ſides into two equal parts, and take away the ſolid angles ſubtended by plane ſuperficial figures, made by the lines coupling thoſe middle ſections; for the Solid remaining after the taking away of thoſe ſolid angles, is called an Icoſidodecaedron. If you divide the ſides of a Cube and of an

Octoedron into two equal parts, and couple the ſections, the ſolid angles ſubtended by the plane ſuperficies made by the coupling-lines, being taken away, there ſhall be left a Solid, which is called an Exoctoedron. So that both of a Dodecaedron, and alſo of an Icoſaedron, the Solid which is made ſhall be called an Icoſidodecaedron; and likewiſe the Solid made of a Cube, and alſo of an Octoedron, ſhall be called an Exoctoedron. But the other Solid, to wit, a Pyramis or Tetraedron, is transformed into a ſimple Solid, for if you divide into two equal parts each of the ſides of the Pyramis, triangles deſcribed of the lines which couple the ſections, and ſubtending and taking away the ſolid angles of the Pyramis, are equal and like unto the equilateral triangles left in each of the baſes, of all which triangles is produced an Octoedron, *viz.* a ſimple, and not a compoſed Solid. For the Octoedron hath four baſes, like in number, form, and mutual inclination with the baſes of the Pyramis, and hath the other four baſes with like ſituation oppoſite and parallel to the former. Wherefore the application of the Pyramis taken twice, maketh a ſimple Octoedron, as the other Solids make a mix'd compound Solid.

## DEFINITIONS.

### I.

*An Exoctoedron is a ſolid figure contained under ſix equal ſquares, and eight equilateral and equal triangles.*

### II.

*An Icoſidodecaedron is a ſolid figure contained under twelve equilateral, equal, and equiangled Pentagons, and twenty equal and equilateral triangles.*

## PROBLEM I. Plate IX. *Fig.* 7.

*To deſcribe an equilateral and equiangled Exoctoedron, and to contain it in a given ſphere, and to prove that the diameter of the ſphere is double to the ſide of the ſaid Exoctoedron.*

*Conſtr.* Suppoſe a Sphere whoſe diameter let be AB, and about the diameter AB let there be deſcribed a ſquare (*a*) and upon the ſquare let there be deſcribed a Cube (*b*) which let be CDEFQTVR; and let the diameter thereof be

a 6. 4.
b 15. 13.

be QR, and the center S. Divide the ſides of the Cube into two equal parts in the points G, H, I, K, L, M, N, O, P, *&c.* and join the middle ſections by the right-lines IN, NO, OP, PI, and ſuch like, which ſubtend the angles of the ſquares or baſes of the Cube; and they are equal (*c*) and contain right-angles, as the angle NIP. c 4. 1.
For the angle NID, which is at the baſe of the Iſoſceles triangle NDI, is the half of a right-angle, and ſo likewiſe is the oppoſite angle RIP. Wherefore the reſidue NIP is a right-angle, and ſo the reſt. Wherefore NIPO is a ſquare. And for the ſame reaſon ſhall the reſt NMLK, KGHI, *&c.* inſcribed in the baſes of the Cube, be ſquares, and they ſhall be ſix in number, according to the number of the baſes of the Cube. Again, foraſmuch as the triangle RIN ſubtendeth the ſolid angle D of the Cube, and likewiſe the triangle KGL the ſolid angle C, and ſo the reſt which ſubtend the right ſolid angles of the Cube, and theſe triangles are equal and equilateral (to wit) being made of equal ſides, and they are the limits or borders of the ſquares, and the ſquares the limits or borders of them; as hath been before proved. Wherefore LMNOPHGK is an Exoctoedron by the definition, and is equilateral, for it is contained by equal ſubtendant lines; it is alſo equiangled, for every ſolid angle thereof is contained under two ſuperficial angles of two ſquares, and two ſuperficial angles of two equilateral triangles.

*Demonſtr.* Foraſmuch as the oppoſite ſides and diameters of the baſes of the Cube are parallels, the plane extended by the right-lines QT and VR, ſhall be a parallelogram. And becauſe alſo in that plane lieth QR, the diameter of the Cube, and in the ſame plane alſo is the line MH, which divideth the ſaid plane into two equal parts, and alſo coupleth the oppoſite angles of the Exoctoedron: this line MH therefore divideth the diameter into two equal parts (*d*); and alſo divideth it ſelf d *cor.* 34. 1.
in the ſame point, which let be S, into two equal parts (*e*). e 4. 1.
And by the ſame reaſon may we prove that the reſt of the lines which join the oppoſite angles of the Exoctoedron, do in S, the center of the Cube, divide one another into two equal parts, for each of the angles of the Exoctoedron are ſet in each of the baſes of the Cube. Wherefore making the center the point S, with the diſtance SH or SM deſcribe a Sphere, and it ſhall touch every one of the angles equidiſtant from the point S.

And becauſe AB the diameter of the ſphere given, is put equal to the diameter of the baſe of the cube, to wit, to the line RT, and the ſame line RT is equal f 33. 3. to the line MH (*f*) which line MH coupling the oppoſite angles of the Exoctoedron, is drawn by the center. Wherefore it is the diameter of the Sphere given which containeth the Exoctoedron.

Laſtly, foraſmuch as in the triangle RFT, the line PO cuts the ſides into two equal parts, it ſhall cut them proportionally with the baſes, to wit, as FR is to g 2. 6. FP, ſo ſhall RT be to PO. (*g*) But FR is double to FP by ſuppoſition: Wherefore RT, or the diameter HM, is alſo double to the line PO, the ſide of the Exoctoedron. Wherefore we have deſcribed, *&c.* Which was required to be done.

## PROBLEM II. Plate IX. *Fig.* 8.

*To deſcribe an equilateral and equiangled Icoſidodecaedron, and to comprehend it in a ſphere given, and to prove that the diameter being divided in extream and mean proportion, maketh the greater ſegment double to the ſide of the Icoſidodecaedron.*

*Conſtr.* Suppoſe that the diameter of the ſphere given a 30. 6. be NL, (*a*) divide the line NL, in extream and mean proportion in the point I, and the greater ſegment there-b 15. 13. of let be NI, and upon the line NI deſcribe a Cube; (*b*) and about this Cube let there be circumſcribed a Dode-c 17. 13. caedron; (*c*) and let the ſame be ABCDEFHKMO, and divide each of the ſides into two equal parts in the points Q, R, S, T, V, X, Y, Z, P, ε, ϑ, G, *&c.* and join the ſections with right-lines, which ſhall ſubtend the angles of the Pentagons, as the lines FG, GV, VQ, QY, YR, RQ, VT, TX, XV, and ſo the reſt.

*Demonſtr.* Foraſmuch as theſe lines ſubtend equal angles of the Pentagons, and thoſe equal angles are contained by equal ſides, to wit, by the half of the ſides of the Pentagons; therefore thoſe ſubtending lines are d 4. 1. equal. (*d*) Wherefore the triangles GQV, YQR, and VXT, and the reſt, which take away ſolid angles of the Dodecaedron, are equilateral.

Again, foraſmuch as in every Pentagon are deſcribed five equal right-lines, coupling the middle ſections of the ſides, as are the lines QV, VT, TS, SR, and RQ, they deſcribe a Pentagon in the plane of the Pentagon of the Dodecaedron. And the ſaid Pentagon is contained in a circle, to wit, whoſe center is the center of a Pentagon of the Dodecaedron. For the lines drawn from that center to the angles of this Pentagon are equal, becauſe they are perpendiculars upon the baſes cut. (*e*) Wherefore e 12. 4. the Pentagon QRSTV, is equiangled. (*f*) And by the f 11. 4. ſame reaſon may the reſt of the Pentagons deſcribed in the baſes of the Dodecaedron, be proved equal and like.

Wherefore thoſe Pentagons are twelve in number: And foraſmuch as the equal and like triangles ſubtend and take away twenty ſolid angles of the Dodecaedron; therefore the ſaid triangles ſhall be twenty in number. Wherefore we have deſcribed an Icoſidodecaedron by the definition, which Icoſidodecaedron is equilateral; for that all the ſides of the triangles are equal and common with the Pentagons; and it is alſo equiangled: For each of the ſolid angles is made of two ſuperficial angles of an equilateral Pentagon, and of two ſuperficial angles of an equilateral triangle.

Now let us prove that it is contained in the given ſphere whoſe diameter is NL. Foraſmuch as perpendiculars drawn from the centers of the Dodecaedron, to the middle ſections of its ſides, are the halfs of the lines which couple the oppoſite middle ſections of the ſides of the Dodecaedron, (*g*) which lines alſo (*h*) do in the center g 3. *cor.* of divide one another into two equal parts. Therefore 17. 13. right-lines drawn from that point to the angles of the h *idem.* Icoſidodecaedron (which are ſet in thoſe middle ſections) are equal; which lines are thirty in number, according to the number of the ſides of the Dodecaedron, for each of the angles of the Icoſidodecaedron are ſet in the middle ſections of each of the ſides of the Dodecaedron. Wherefore making the center of the Dodecaedron, and the ſpace, any one of the lines drawn from the center to the middle ſections, deſcribe a ſphere, and it ſhall paſs through all the angles of the Icoſidodecaedron, and ſhall contain it.

And foraſmuch as the diameter of this ſolid, is that right-line whoſe greater ſegment is the ſide of the Cube inſcribed in the Dodecaedron, (*i*) which ſide is NI by i 4. *cor.* ſuppoſition. Wherefore that ſolid is contained in the 17. 13. ſphere given, whoſe diameter is put to be the line NL.

Now let us prove that the great ſegment of the diameter is double to QV the ſide of the ſolid. Foraſmuch as the ſides of the triangle AEB are in the points Q and V divided into two equal parts, the lines QV and BE are parallels. (*k*) Wherefore as AE is to AV, ſo is EB to VQ. k *cor.* 39. 1.

(*l*) But

l 2. 6. (*l*) But the line AE is double to the line AV. Wherefore the line BE is double to the line QV. (*m*) Now m 4. 6. the line BE is equal to NI, or to the ſide of the Cube; (*n*) which line NI is the greater ſegment of the diameter n 2. *cor.* of 17. 13. NL. Wherefore the greater ſegment of the diameter given is double to the ſide of the Icoſidodecaedron inſcribed in the given ſphere. Wherefore we have deſcribed, &c. Which was required to be done.

## *ADVERTISEMENT.*

To the underſtanding of the nature of this Icoſidodecaedron, you muſt well conceive the paſſions and proprieties of both theſe ſolids, of whoſe baſes it conſiſteth, to wit, of the Icoſaedron and of the Dodecaedron. And altho' in it the baſes are placed oppoſitely, yet have they to one another one and the ſame inclination. By reaſon whereof there lie hidden in it the actions and paſſions of the other regular Solids. And I would have thought it not impertinent to the purpoſe to have given the inſcriptions and circumſcriptions of this Solid, if want of time had not hindered. But that the Reader may the better attain to the underſtanding thereof, I have here following briefly ſhewn, how it may in or about every one of the five regular Solids be inſcribed or circumſcribed; by the help whereof he may, with ſmall labour, or rather none at all, having well poiſed and conſidered the Demonſtrations appertaining to the foreſaid five regular Solids, demonſtrate both the inſcription of the ſaid Solids in it, and the Inſcription of it in the ſaid Solids.

### *Of the Inſcriptions and Circumſcriptions of an Icoſidodecaedron.*

An Icoſidodecaedron may contain the other five regular bodies. For it will receive the angles of a Dodecaedron in the centers of the triangles which ſubtend the ſolid angles of the Dodecaedron, which ſolid angles are twenty in number, and are placed in the ſame order in which the ſolid angles of the Dodecaedron, taken away, or ſubtended by them, are. And for that reaſon it ſhall receive a Cube and a Pyramis contained in the Dodecaedron, when as the angles of the one are ſet in the angles of the other.

An Icoſidodecaedron receiveth an Octoedron, in the angles cutting the ſix oppoſite ſections of the Dodecaedron, even as if it were a ſimple Dodecaedron.

And it containeth an Icoſaedron, placing the twelve angles of the Icoſaedron in the ſame centers of the twelve Pentagons.

It may alſo by the ſame reaſon be inſcribed in each of the five regular bodies, to wit, in a Pyramis, if you place four triangular baſes concentrical with four baſes of the Pyramis, after the ſame manner that you inſcribed an Icoſaedron in a Pyramis; ſo likewiſe may it be inſcribed in an Octoedron, if you make eight baſes thereof concentrical with the eight baſes of the Octoedron. It ſhall alſo be inſcribed in a Cube, if you place the angles which receive the Octoedron in it, in the centers of the baſes of the Cube. Again, you ſhall inſcribe it in an Icoſaedron, when the triangles formed of the Pentagon baſes, are concentrical with the triangles which make a ſolid angle of the Icoſaedron.

Laſtly, it ſhall be inſcribed in a Dodecaedron, if you place each of the angles thereof in the middle ſections of the ſides of the Dodecaedron, according to the order of its conſtruction.

The oppoſite plain ſuperficies alſo of this ſolid are parallels. For the oppoſite ſolid angles are ſubtended by parallel plain ſuperficies, as well in the angles of the Dodecaedron ſubtended by triangles, as in the angles of the Icoſaedron ſubtended by Pentagons, which thing may eaſily be demonſtrated. Moreover, in this ſolid are infinite properties and paſſions, ſpringing from the ſolids whereof it is compoſed.

Wherefore it is manifeſt, that a Dodecaedron and an Icoſaedron mixed, are transformed into one and the ſelf ſame ſolid of an Icoſidodecaedron. A Cube alſo and an Octoedron are mixed and altered into another ſolid, to wit, into one and the ſame Exoctoedron. But a Pyramis is transformed into a ſimple and perfect ſolid, to wit, into an Octoedron.

If we will frame theſe two ſolids joined together into one ſolid, this only muſt we obſerve.

In the Pentagon of a Dodecaedron inſcribe a like Pentagon, and let its angle be ſet in the middle ſections of the Pentagon circumſcribed, and then upon the ſaid Pentagon inſcribed, let there be ſet a ſolid angle of an Icoſaedron, and ſo obſerve the ſame order in each of the baſes of the Dodecaedron, and the ſolid angles of the Icoſaedron, ſet upon theſe Pentagons, ſhall produce a ſolid conſiſting of the whole Dodecaedron, and whole Icoſaedron. In like manner, if in every baſe of the Icoſaedron

ſaedron, the ſides being divided into two equal parts, be inſcribed an equilateral triangle, and upon each of thoſe equilateral triangles be ſet a ſolid angle of a Dodecaedron, there ſhall be produced the ſame ſolid conſiſting of the whole Icoſaedron, and of the whole Dodecaedron.

And after the ſame order, if in the baſes of a Cube be inſcribed ſquares ſubtending the ſolid angles of an Octoedron, or if in the baſes of an Octoedron be inſcribed equilateral triangles ſubtending the ſolid angles of a Cube, there ſhall be produced a ſolid conſiſting of either of the whole ſolids, to wit, of the whole Cube, and of the whole Octoedron.

But equilateral triangles inſcribed in the baſes of a Pyramis, having their angles ſet in the middle ſections of the ſides of the Pyramis, and the ſolid angles of a Pyramis, ſet upon the ſaid equilateral triangles, there ſhall be produced a ſolid conſiſting of two equal and like Pyramids.

And now if in theſe ſolids thus compoſed, you take away the ſolid angle, there ſhall be reſtored again the firſt compoſed ſolids, to wit, the ſolid angles taken away from a Dodecaedron and an Icoſaedron compoſed into one, there ſhall be left an Icoſidodecaedron, the ſolid angles taken away from a Cube and an Octoedron compoſed into one ſolid, there ſhall be left an Exoctoedron. Moreover, the ſolid angles taken away from two Pyramids compoſed into one ſolid, there ſhall be left an Octoedron.

### *Of the nature of a trilateral and equilateral Pyramis.*

1. A trilateral equilateral Pyramis is divided into two equal parts, by three equal ſquares, which in the center of the Pyramis cut one another into two equal parts, and perpendicularly, and whoſe angles are ſet in the middle ſections of the ſides of the Pyramis.

2. From a Pyramis are taken away four Pyramids like unto the whole, which utterly take away the ſides of the Pyramis, and that which is left is an Octoedron, inſcribed in the Pyramis, in which all the ſolids inſcribed in the Pyramis are contained.

3. A perpendicular drawn from the angle of the Pyramis to the baſe, is double to the diameter of the Cube inſcribed in it.

4. And a right-line joining the middle ſections of the oppoſite ſides of the Pyramis is triple to the ſide of the ſame Cube.

5. The ſide alſo of a Pyramis is triple to the diameter of the baſe of the Cube.

6. Wherefore the ſame ſide of the Pyramis is in power double to the right-line which joins the middle ſection of the oppoſite ſides.

7. And it is in power ſeſquialter to the perpendicular which is drawn from the angle to the baſe.

8. Wherefore the perpendicular is in power ſeſquitertia to the line which joins the middle ſections of the oppoſite ſides.

9. A Pyramis and an Octoedron inſcribed in it, alſo an Icoſaedron inſcribed in the ſame Octoedron, contain one and the ſame ſphere.

### *Of the nature of an Octoedron.*

1. Four perpendiculars of an Octoedron, drawn in four baſes thereof from two oppoſite angles of the ſaid Octoedron, and joined together by thoſe four baſes, deſcribe a Rhombus, or Diamond figure; one of whoſe diameters is in power double to the other diameter.

2. For it hath the ſame proportion that the diameter of the Octoedron hath to the ſide of the Octoedron.

3. An Octoedron and an Icoſaedron inſcribed in it, do contain one and the ſame ſphere.

4. The diameter of the ſolid of the Octoedron is in power ſeſquialter to the diameter of the circle which containeth the baſe, and is in power duple ſuperbipartiens tertias (that is, as 8 to 3,) to the perpendicular or ſide of the foreſaid Rhombus; and moreover is in length triple to the line which joins the centers of the next baſes.

5. The angle of the inclination of the baſes of the Octoedron, doth, with the angle of the inclination of the baſes of the Pyramis, make angles equal to two right-angles.

### *Of the nature of a Cube.*

1. The diameter of a Cube is in power ſeſquialter to the diameter of his baſe.

2. And is in power triple to his ſide.

3. And unto the line which joins the centers of the next baſes, it is in power ſextuple.

4. Again, the ſide of the Cube, is to the ſide of the Icoſaedron inſcribed in it, as the whole is to the greater ſegment.

5. Unto the ſide of the Dodecaedron, it is as the whole is to the leſſer ſegment.

6. Unto the ſide of the Octoedron it is in power duple.

7. Unto the ſide of the Pyramis it is in power ſubduple.

8. Again, the Cube is triple to the Pyramis, but to the Cube the Dodecaedron is in a manner double. Wherefore the ſame Dodecaedron is in a manner ſextuple to the ſaid Pyramis.

### *Of the nature of the Icoſaedron.*

1. Five triangles of an Icoſaedron, make a ſolid angle, the baſes of which triangles make a Pentagon. If therefore from the oppoſite baſes of the Icoſaedron be taken the other Pentagon by them deſcribed; theſe Pentagons ſhall in ſuch ſort cut the diameter of the Icoſaedron which joins the foreſaid oppoſite angles, that that part which is contained between the planes of theſe two Pentagons ſhall be the greater ſegment, and the reſidue which is drawn from the plane to the angle, ſhall be the leſſer ſegment.

2. If the oppoſite angles of two baſes joined together, be coupled by a right-line, the greater ſegment of that right-line is the ſide of the Icoſaedron.

3. A line drawn from the center of the Icoſaedron to the angles, is in power quintuple to half that line which is taken between the Pentagons, or of the of half that line which is drawn from the center of the circle which containeth the foreſaid Pentagon, which two lines are therefore equal.

4. The ſide of the Icoſaedron containeth in power either of them, and alſo the leſſer ſegment, to wit, the line which falleth from the ſolid angle to the Pentagon.

5. The diameter of the Icoſaedron containeth in power the whole line, which coupleth the oppoſite angles of the baſes joined together, and the greater ſegment thereof, to wit, the ſide of the Icoſaedron.

6. The diameter alſo is in power quintuple to the line which was taken between the Pentagons, or to the line which is drawn from the center to the circumference of the circle which containeth the Pentagon compoſed of the ſides of the Icoſaedron.

7. The dimetient containeth in power the right-line which coupleth the centers of the oppoſite baſes of the Icoſaedron, and the diameter of the circle which containeth the baſe.

8. Again, the ſaid dimetient containeth in power the diameter of the circle which containeth the Pentagon, and alſo the line which is drawn from the center of the ſame circle to the circumference: that is, it is quintuple to the line drawn from the center to the circumference.

9. The line which coupleth the centers of the oppoſite baſes containeth in power the line which coupleth the centers of the next baſes, and alſo the reſt of that line of which the ſide of the Cube inſcribed in the Icoſaedron is the greater ſegment.

10. The line which coupleth the middle ſections of the oppoſite ſides, is triple to the ſide of the Dodecaedron inſcribed in it,

11. Wherefore, if the ſide of the Icoſaedron and the greater ſegment thereof be made one line, the third part of the whole is the ſide of the Dodecaedron inſcribed in the Icoſaedron.

### *Of the Dodecaedron.*

1. The diameter of a Dodecaedron containeth in power the ſide of the Dodecaedron, and alſo that right-line to which the ſide of the Dodecaedron is the leſſer ſegment, and the ſide of the Cube inſcribed in it is the greater ſegment, which line is that which ſubtendeth the angle of the inclination of the baſes, contained under two perpendiculars of the baſes of the Dodecaedron.

2. If there be taken two baſes of the Dodecaedron, diſtant from one another by the length of one of the ſides, a right-line coupling their centers being divided in extreme and mean proportion, maketh the greater ſegment the right-line which coupleth the centers of the next baſes.

3. If by the centers of five baſes ſet upon one baſe, be drawn a plain ſuperficies, and by the centers of the baſes which are ſet upon the oppoſite baſe, be drawn alſo a plain ſuperficies, and then be drawn a right-line, coupling the centers of the oppoſite baſes, that right-line is ſo cut, that each of his parts ſet without the plain ſuperficies, is the greater ſegment of that part which is contained between the planes.

4. The ſide of the Dodecaedron is the greater ſegment of the line which ſubtendeth the angle of the Pentagon.

5. A perpendicular-line drawn from the center of the Dodecaedron to one of the baſes, is in power quintuple to half the line which is between the planes.

6. And therefore the whole line which coupleth the centers of the oppoſite baſes is in power quintuple to the whole line which is between the ſaid planes.

7. The line which ſubtendeth the angle of the baſe of the Dodecaedron, together with the ſide of the baſe, are in power quintuple to the line which is drawn from the center of the circle which containeth the baſe, to the circumference.

8. A ſection of a ſphere containing three baſes of the Dodecaedron, taketh a third part of the diameter of the ſaid ſphere,

9. The ſide of the Dodecaedron and the line which ſubtendeth the angle of the Pentagon, are equal to the right-line which coupleth the middle ſections of the oppoſite ſides of the Dodecaedron.

# THE

# THEOREMS

# OF

# ARCHIMEDES.

Concerning the Sphere and Cylinder, inveſtigated by the *Method of indiviſibles*, and briefly demonſtrated, by the Reverend and Learned Dr. ISAAC BARROW.

THE main Deſign of *Archimedes* in his *Treatiſe of the Sphere and Cylinder*, is to reſolve theſe four Problems.

1. *To find the proportion of the ſuperficies of a ſphere to any determinate circle; or to find a circle equal to the ſuperficies of a given ſphere.*

2. *To find the proportion of the ſuperficies of any ſegment of a ſphere to any determined circle; or to find a circle equal to the ſuperficies of any aſſigned ſegment.*

3. *To find the proportion of the ſphere it ſelf (or of its ſolid content) to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given ſphere.*

4. *To find the proportion of a ſegment of a ſphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given ſegment.*

Theſe four Problems *Archimedes* proſecutes ſeparately, and lays down Theorems immediately ſubſervient to their ſolution; but we reduce them to two: For ſince an Hemiſphere is the ſegment of a ſphere, and the method of finding out its relations, in reſpect to the ſuperficies and ſolid content, is comprehended in the general method of inveſtigating the proportion of the ſegments: And from the ſuperficies and ſolid content of an Hemiſphere already found, the double of them, (that is the ſuperficies and content of the whole ſphere) is at the ſame time given. And indeed 'tis ſuperfluous and foreign from the Laws of good Method, to inveſtigate their relations diſtinctly and ſeparately; ſo that if it were not a crime, I might on this account blame even *Archimedes* himſelf.

The whole matter therefore is reduc'd to theſe two Problems.

1. *To find the proportion of the ſuperficies of any ſegment of a ſphere to a determinate circle; or to find a circle equal to the ſuperficies of a given ſegment.*

2. *To find the proportion of the ſolidity of any ſegment of a ſphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to an aſſign'd ſegment of a ſphere.*

I ſhall reſolve theſe Problems by another much eaſier and ſhorter method: In which the order being inverted, firſt, I ſhall ſeek the ſolidity of a ſegment, and from thence deduce its ſuperficies; a thing which is in my judgment well worth obſerving, and perform'd (as I know of) by none.

Firſt therefore, for finding the ſolidity of a ſegment, I ſhall lay down two, commonly known and receiv'd, Suppoſitions, *viz.*

1. *That a ſeries of magnitudes proceeding in Arithmetical Progreſſion from nothing (incluſive) or whoſe common difference is equal to the leaſt magnitude, is ſubduple of as many quantities equal to the greateſt:* (i. e. *ſubduple of the product of the greateſt term and number of terms:*) So that if the *ſum of the terms* be called *z*, the *greateſt term g*, and the *number of terms n*, then will

$$z = \frac{ng}{2}, \text{ or } 2z = ng.$$

The truth of this Propoſition will eaſily appear by expreſſing the ſeries twice, and inverting the order;

$o, a, 2a, 3a, 4a.$
$4a, 3a, 2a, a, o.$

For ſo the difference always being equal to the leaſt quantity, 'twill be evident that each two correſpondent terms taken together are equal to the greateſt term; and alſo, that the ſeries taken twice is equal to the greateſt term repeated as many times as there are terms, *i. e.* the laſt term drawn into the number of terms.

We have in a *triangle* a very clear and eaſy example of this moſt uſeful Propoſition, which is prov'd hence, *to be half a parallelogrom having the ſame altitude, and ſtanding on the ſame baſe.*

Suppoſe the altitude AE (Plate IX. *Fig.* 9.) of the triangle AEZ to be divided into parts indefinitely many and ſmall, AB, BC, CD, DE, and parallels BZ, CZ, DZ, EZ, drawn thro' the points of Diviſion; all theſe proceed from nothing in an Arithmetical progreſſion, and conſequently the ſum of them all, (that is, the triangle AEZ) is ſubduple of the greateſt EZ drawn into the altitude AE, by which the ſum of the terms is expreſs'd, that is ſubduple of the Parallelogram EY, whoſe baſe is EZ, and altitude AE.

But the illuſtration of the Rule will conduce more to our deſign by inferring hence, *That a circle is equal to half of the radius drawn into the circumference,* after this manner. Conceive a circle to conſiſt of as many concentric Peripheries as there are points or equal parts indefinitely many and ſmall in the radius. Theſe Peripheries, as well as their radii, proceed from the center or nothing in an Arithmetical progeſſion; and therefore their ſum, that is, the whole circle is equal to half the greateſt (or extreme circumference) drawn into the number of terms, that is, the radius.

After the ſame manner we may ſuppoſe the ſector AEZ (*Fig.* 10.) to conſiſt of as many concentric Arcs BZ, CZ, DZ, EZ, as there are points (or equal parts indefinitely ſmall) in the radius AE, which Arcs, as their radii, proceeding from a point or nothing in an Arithmetical progreſſion, the ſector alſo will be equal to half the radius drawn into the extreme Arc EZ. Which may be made evident alſo after this manner: Let us ſuppoſe the right line EY to be perpendicular to the radius AE, draw the right-line AY, and from the points B, C, D, of diviſion in the radius, draw BY, CY, DY, parallel to EY, and terminated at AY. Becauſe EY : DY ( : : *rad.* AE : *rad.* AD) : : *Arc.* EZ : *Arc.* DZ, and EY = EZ, then will DY = Arc. DZ; and in like manner will CY = CZ, and BY = BZ. Whence the triangle AEY Will be = to the ſector AEZ, that is, $\frac{AE\times EY}{2}\left(\frac{AE\times EZ}{2}\right)=$ ſector AEZ. By this means we collect that celebrated Theorem of *Archimedes, that a circle is equal to a triangle whoſe baſe is equal to the radius, and altitude equal to the periphery of the circle*; and that without any inſcription or circumſcription of figures, by only ſuppoſing that the Area or Superficies of the circle conſiſts of infinitely many concentric Peripheries. Which *method of Indiviſibles*, (now firſt of all known to me) ſeems no leſs evident (nay more evident) and perhaps leſs fallacious than that wherein planes are ſuppoſed to conſiſt of parallel right-lines, and ſolids of parallel planes; as hereafter ſhall be evident, when we ſhall collect, by this method, the proportions of ſpheric and cylindric ſuperficies to one another, by knowing the ſolid content; and on the other hand, the ſolid content, by knowing the ſuperficies, with admirable facility, and full ſatisfaction in thoſe things which are with difficulty obtained by pure *Geometry*.

*Let us ſuppoſe a ſeries of quantities to proceed from* 0 (*incluſive*) *in a duplicate Arithmetic proportion, that is,* 0, 1, 4, 9, 16, &c. *the ſquares of numbers in a ſimple Arithmetic progreſſion,* 0, 1, 2, 3, 4, &c. *And the triple of this ſeries will always exceed the greateſt term multiplied by the number of terms; but the number of terms increaſing, the proportion continually approximates, till at laſt it comes to an equality, when the number of terms is increaſed* in infinitum.

$$\begin{array}{r} 3\times\overline{0+1}=3.\\ 2\times 1=2.\end{array}\quad \frac{3}{2}$$

$$\begin{array}{r} 3\times\overline{0+1+4}=15.\\ 3\times 4=12.\end{array}\quad \frac{15}{12}=\frac{5}{4}$$

$$\begin{array}{r} 3\times\overline{0+1+4+9}=42.\\ 4\times 9=36.\end{array}\quad \frac{42}{36}=\frac{7}{6}$$

$$\begin{array}{r} 3\times\overline{0+1+4+9+16}=90.\\ 5\times 16=80.\end{array}\quad \frac{90}{80}=\frac{9}{8}$$

$$\begin{array}{r} 3\times\overline{0+1+4+9+16+25}=165.\\ 6\times 25=150.\end{array}\quad \frac{165}{150}=\frac{11}{10}$$

As for example, if the terms are two, the triple of the terms will be to the greateſt term drawn into the number of terms as 3 to 2; if there be three terms as 5 to 4; if four, as 7 to 6; if five, as 9 to 8, and ſo continually: So that the antecedents of theſe proportions always mutually exceed one another by the number 2; and ſo every antecedent its conſequent by 1. Whence it is evident that by how much the greater the number of terms is, by ſo much the more the proportion tends to equality. So 100 to 99 is leſs diſtant from the proportion of equality than 10 to 9. From hence, ſuppoſing the number of terms infinite (or infinitely great,) the triple of quantities proceeding thus in a duplicate proportion (or as the ſquares of the numbers, 0, 1, 2, 3, 4, &c.) will be equal to as many quantities equal to the greateſt term.

The ſame, as to the ſubſtance of it, is laid down by *Archimedes* in his *Book of Spirals*, as the Foundation of many Argumentations, in that, and other Books, and is well demonſtrated by our learned Country-man Dr. *Wallis:* However, I thought fit to illuſtrate the matter by this method, as being not unworthy our Conſideration, and very perſpicuous and intelligible in this, that 'tis free from Fractions: And by the way 'tis obſerv'd, that from hence we may eaſily find the proportion of a ſeries triple to as many terms equal to the greateſt, *viz.* as twice the number of terms leſs one, to twice the number of terms leſs two. So that if the number of the terms be 6, the proportion of a ſeries triple to as many terms equal to the greateſt will be as 11, It will be a very eaſy and apt Illuſtration of this Rule, if we infer hence, *That a Cone is ſubtriple of a Cylinder, having an equal baſe and altitude.* For let us ſuppoſe the altitude AE (*Fig.* 11.) of the Cone ZY to be divided into equal and indefinitely number of parts, by as many parallel right-lines ZY, and the lines ZY will be as the numbers 1, 2, 3, 4, *&c.* and the ſquares or circles conſtituted upon the diameters ZY, as 1, 4, 9, 16, *&c.* whence all thoſe circles, or the whole Cone AZY (made up of the ſame) will be ſubtriple of as many circles equal to the greateſt, conſtituted on the greateſt diameter ZEY, that is, ſubtriple of a cylinder whoſe baſe is AEY, and altitude AE.

There occur two other moſt apt examples of this Rule, *viz.* by inferring, *That the complement of a Semiparabola is ſubtriple of a parallelogram having the ſame baſe and height*; as alſo, *That the ſpace comprehended by the Spiral and Radius is ſubtriple of the circle in which the Spiral is generated:* But of theſe in another place. Wherefore to go on with what we began, theſe two Rules being ſuppoſed; let us conceive ZAY (*Fig.* 12.) to be a ſegment of a ſphere, X its center, AT its diameter, and ZAYT a great circle paſſing thro' the vertex, and the part AE of the Axe to be divided into an indefinitely number of equal parts; and let us imagine parallel-lines to be drawn thro' the points of diviſion generating circles in the ſphere, whoſe Radii let be BZ, CZ, DZ, and diameters ZY. I ſuppoſe the ſegment of a ſphere to conſiſt of all theſe parallel circles, whoſe number is as great as that of the points, or equal indefinitely ſmall parts in the Axe AE, according to the known *Method of Indiviſibles.*

But nor for brevity's ſake, let the diameter AT be called $d$, and the radius of the ſphere $r$, (if need be) and the Axe AE, by which the number of terms is expreſs'd, call $n$, and one of the equal parts $a$; which being ſettled, 'tis evident, (*by the Elements*) that $\overline{BZ}^2 = AB \times BT = a \times \overline{d - a} = ad - a^2$, and in like manner $CZ^2 = AC \times CT = 2a \times \overline{d - 2a} = 2ad - 4a^2$, and by the ſame reaſoning $DZ^2 = AD \times DT = 3ad - 9a^2$, and $EZ^2 = AE \times ET = 4ad - 16a^2$, &c. that is, that the ſquares of the radii of the circles ZY are to one another as the rectangles, $ad$, $2ad$, $3ad$, $4ad$, &c. (which proceed in an Arithmetical Progreſſion from 0) leſs by the ſquare $a^2$, $4a^2$, $9a^2$, $16a^2$, &c. which go on as the ſquares of the number, 1, 2, 3, 4, &c. But by our firſt Rule, all the Rectangles 0, $ad$, $2ad$, $3ad$, $4ad$, &c. are equal to half as many terms equal to the greateſt AE × AT or $nd$, that is, $=\frac{nd \times n}{2}$.

Moreover, by our ſecond Rule, all the ſquares 0, $a^2$, $4a^2$, $9a^2$, $16a^2$, &c. taken together, are equal to a third part of as many terms equal to the greateſt $\overline{AE}^2$ or $n^2$, that is, $=\frac{n^2 \times n}{3}$.

Wherefore all the ſquares deſcribed upon the radii BZ, CZ, DZ, EZ, conjunctly, are equal to the difference $\frac{nnd}{2} - \frac{nnn}{3}$, (or the terms being reduc'd to the ſame denomination,) $\frac{3ndn - 2\,nnn}{6}$ and their quadruple, that is, all the ſquares deſcribed upon the diameters ZY, are equal to $\frac{12\,ndn - 8\,n^3}{6}$ or $\frac{6\,ndn - 4\,n^3}{3}$ whence a ſegment of a ſphere is equal to a Cylinder, the diameter of whoſe baſe is the ſide of a ſquare equal to $6\,nd - 4n^2$, and altitude is $\frac{1}{3}n$; or to a Cone having the ſame baſe, but the altitude $n$, or which is all one, having a baſe whoſe radius is $\sqrt{\frac{6\,nd - 4\,n^2}{4}}$ or $\sqrt{\frac{3}{2}nd - n^2}$, and altitude $n$ as before. Which Cone we may change into a Cone upon the ſame baſe ZY with the ſegment ZAY, by ſaying, as $ZE^2$ (*i. e.* $dn - n^2$) to $\frac{3}{2}nd - n^2$ or (both terms being divided by $n$) as $d - n$ to $\frac{3}{2}d - n$, ſo is reciprocally $n$ to the altitude of the Cone ſought: Or in the figure by making, *as* TE *to* TE + XA, *ſo is* EA *to* ES. *For* ES *will be the altitude of the Cone* ZSY *equal to the ſegment of the ſphere* ZAY. Which is that noted Theorem of *Archimedes*, demonſtrated by him with ſo much labour and prolixity.

Hence, if the given ſegment be a *Hemiſphere*, and ſo $n = \frac{1}{2} d$ or $r$, then $d$ or $2r$ will be the altitude of a *Cone*, which having a baſe equal to the baſe of the *Hemiſphere* (or to the greateſt circle in the ſphere) will be equal to the *Hemiſphere*. And a *Cone* whoſe baſe is double of the greateſt circle, and the altitude $2r$, or the *Cylinder* whoſe baſes is $\frac{1}{3}$ of the greateſt circle, and altitude $2r$ will be equal to the whole *Sphere*. Whence the whole *Sphere* is $\frac{2}{3}$ of a *Cylinder*, the diameter of whoſe baſe is $2r$, and the altitude alſo $2r$. And this is the chief Theorem of *Archimedes*, viz. *That a ſphere is ſubſeſquialter or $\frac{2}{3}$ of that Cylinder, whoſe Altitude and Diameter of the baſe is equal to the Diameter of the Sphere.*

Furthermore, not to paſs over any thing in our Author which ſeems to be to our purpoſe:

If to the ſum firſt found, repreſenting a ſegment, *viz.* $\frac{6ndn - 4nnn}{3}$, we add $\frac{2d^2n - 6dn^2 + 4n^3}{3}$ $\left(= \frac{4dn - 4n^2}{3} \times \frac{d - 2n}{2} = \frac{4}{3} ZE^2 \times XE\right)$ repreſenting the Cone ZXY, the aggregate $\frac{2}{3}\, d\,d\,n$ will repreſent the Sector of the Sphere ZXYA, which for that reaſon will be equal to a Cylinder, the diameter of whoſe baſe is $\surd\, dn$, and the altitude $\frac{2}{3} d$, or to a Cone, the diameter of whoſe baſe is $\surd\, dn$, and the altitude $2d$, or alſo to a Cone, the Radius of whoſe baſe is $\surd\, dn$, and the altitude $\frac{1}{2} d = r$ (it being reciprocally as $4dn : dn :: 2d : \frac{1}{2} d$,) that is, to a Cone, the Radius of whoſe baſe is the Line AZ, drawn from the vertex to the circumference of the baſe of the ſegment, (for $AZ^2 = TA \times AE = dn$,) and the altitude $r$. And this is the next famous Theorem of *Archimedes*, concerning the ſolidity of the ſector of the Sphere, *viz. That the ſector of a ſphere is equal to a Cone, whoſe baſe is a circle deſcribed by a Radius equal to a line drawn from the vertex to the circumference of the baſe of the ſegment, and whoſe altitude is equal to the Radius of the ſphere.*

And thus I think I have compleated that which belongs to the ſolidity of a ſphere, and its parts, with ſufficient brevity and perſpicuity. From hence we ſhall deduce the Reſolution of the other Problem, which I propoſed concerning the ſurface of the ſegment of a ſphere; and then of the whole ſphere. To obtain this, as we ſuppoſed before, a *Circle* to conſiſt of concentric

Peripheries, and the *Sector of a Circle* of concentric Arcs, (in the number of which, the greateſt, and the leaſt, or a point is reckon'd: So now we ſuppoſe ſpheres to conſiſt of concentric ſpherical ſuperficies, and the *Sector of Spheres* of like concentric ſuperficies; as for example, the ſector of the ſphere ZAE (*Fig.* 13.) of the ſuperficies BZ, CZ, DZ, EZ, *&c.*) which ſuppoſition indeed ſeems ſo eaſy and natural, that in my judgment 'tis ſufficient only to propoſe it; neither is a further explication wanting to gain an aſſent to it.

2. We ſuppoſe theſe ſpherical ſuperficies to be in a duplicate Ratio of the Radius of the ſpheres: This is the common affection of all like ſuperficies, and it ſeems to agree very well with the ſuperficies of ſpheres, becauſe they appear to be moſt uniform and ſimilar. But this Suppoſition might eaſily be evinc'd and eſtabliſh'd by the ſame ſort of arguing, as ſpheres are proved to be in triplicate proportion to their Diameters or Radii; or might have been join'd as a *Corollary* to *Prop.* 17. and 18. *Elem.* 12. where the ſuperficies of like Polygons are ſuppos'd to be inſcribed in ſpheres, having as well the ſuperficies in a duplicate, as the ſolidity in a triplicate Ratio of the Diameters of the Spheres. Theſe things being premis'd let us ſuppoſe AE a Radius, or the ſide of the Sector of a Sphere EAZ, to be divided into equal and indefinitely many ſmall parts, and the ſector AEZ to conſiſt of theſe ſpherical ſuperficies BZ, CZ, DZ, EZ, it will be evident that all thoſe ſuperficies in the Progreſſion are as the ſquares of the Radii, that is, as $\overline{AB}^2$, $\overline{AC}^2$, $\overline{AD}^2$, $\overline{AE}^2$, *&c.* or as the ſquares of the numbers 1, 2, 3, 4, *&c.* whence by our ſecond Rule, the ſum of all theſe ſuperficies, that is, the ſector AEZ, will be $\frac{1}{3}$ of as many ſuperficies equal to the greateſt EZ, that is, $\frac{1}{3}$ of the greateſt EZ, drawn into $r$ the number of terms. Whence a ſector is equal to a Cylinder, whoſe baſe is $\frac{1}{3}$ of the greateſt or extreme ſuperficies of the ſector, and whoſe altitude is $r$: Or to a Cone whoſe baſe is equal to the ſuperficies of the ſector, and its altitude $r$, which is the laſt of *Lib.* 1. but we juſt now prov'd that a ſector is equal to a Cone whoſe altitude is $r$, and baſe a circle, deſcrib'd by the Radius YE, drawn from the vertex of the ſegment EYZ to the circumference of the baſe. Wherefore a Cone, whoſe altitude is $r$, and baſe equal to the ſuperficies of the ſector, is equal to a Cone of the ſame altitude, whoſe baſe is a circle deſcrib'd by the Radius YE.

And ſo the ſuperficies of the ſector EYZ is equal to a circle deſcrib'd by the Radius YE. Which certainly is the principal Theorem of all thoſe that occur in the Books of *Archimedes*, nor is there found a more excellent one in all Geometry; *viz. That the ſuperficies of any ſegment of a ſphere is equal to a circle whoſe Radius is a right-line drawn from the vertex of the ſegment to the circumference of the baſes:* And hence, *that the ſuperficies of an Hemiſphere is double to the baſe, or equal to two great cirles of the ſphere.*

For in this Caſe $\overline{YE^2} = \overline{AZ^2} + \overline{AY^2} = \overline{2 AE^2}$, and conſequently a circle deſcribed by the Radius YE (*Fig.* 14.) is equal to two circles deſcrib'd by the Radius AE. Whence alſo, *the ſuperficies of the whole ſphere is quadruple, a circle having the ſame Radius with the ſphere*, that is, *quadruple the greateſt circle in the ſphere; and equal to a circle whoſe Radius is the diameter of the ſphere.* From hence it follows, *that the ſuperficies of a ſphere is equal to the ſuperficies of a Cylinder of the ſame height and breadth*; for the ſuperficies of that Cylinder is quadruple to the baſe as we ſhall ſhew hereafter. And theſe are the moſt noted Theorems of *Archimedes*. Nay, from hence all thoſe things follow, which he has written concerning the ſuperficies of ſpheres, and their ſegments. So that from theſe few and eaſy Suppoſitions, I have demonſtrated whatever ſeems to be of any Note in the Books *of the Sphere and Cylinder*.

I will only add, that after by the method of *Archimedes*, (for I think ſcarce any other can be invented, beſides ours, for finding the ſolidity) the ſuperficies of ſegments are found equal to the circle deſcribed by the Radii YE; hence it will plainly follow, that the ſuperficies of ſpheres, and thence of like ſectors are in a duplicate ratio of the Radii of the ſpheres; and conſequently from the ſuperficies thus found, the contents of ſegments, and of whole ſpheres may be mutually deduced, and that very clearly and expeditiouſly, after this manner. Becauſe in the ſector EAZ (*Fig.* 13.) the ſuperficies BZ, CZ, DZ, EZ, proceed as the ſquares deſcrib'd upon AB, AC, AD, AE, that is, as 1, 4, 9, 16, &c. the whole ſector will be equel to $\frac{1}{3}$ of as many ſuperficies equal to the greateſt EZ, or $\frac{1}{3}$ EZ × *r*, that is, to a Cylinder whoſe baſe is $\frac{1}{3}$ EZ, and altitude *r*, or to a Cone whoſe baſe is EZ, and altitude *r*. But EZ is ſuppoſed equal to a circle whoſe Radius is YE, wherefore the ſector EAZ is equal to a Cone whoſe baſe is a circle deſcribed by the Radius YZ and altitude *r*: Which is *Archimedes*'s univerſal Theorem for the contents of *Sectors*. Whence if from this the Cone ZAE ſtanding on the baſe of the ſegment EYZ, and having the vertex at the center of the ſphere A, be ſubducted, you'll have that ſegment EYZ. But when the ſector EYZ is a Hemiſphere, there will be no ſuch Cone to be ſubducted; and for that reaſon a Cylinder whoſe baſe is $\frac{2}{3}$ EZ, and altitude *r*, or the Cone whoſe baſe is 2 EZ, and altitude likewiſe *r*, will be equal to the whole ſphere. But the Superficies of the Hemiſphere EZ, is proved to be equal to two of the greateſt circles in the ſphere, whence the whole ſphere is given. This is *Archimedes*'s firſt and principal Theorem, for the content of a ſphere; whence 'tis eaſily deduced, that a ſphere is $\frac{2}{3}$ of a circumſcrib'd Cylinder, that is, of a Cylinder whoſe altitude and diameter of its baſe is equal to the diameter of the ſphere.

The Doctrine of our author [*Archimedes*] ſeems to makes againſt, and ſubvert the new and celebrated *Method of indiviſibles*, and is preſs'd to that end by *Tacquet*; for inſtance, (*Prop.* 2. *lib.* 1. *Cylindr.*) For the uſual proceſs of that method ſeems to exhibit the dimenſions of the ſuperficies of a Cone, (as alſo of a ſphere, and of other Curves) different enough from what our Author and others have demonſtrated: As for example, let us ſuppoſe ABCD (*Fig.* 15.) a right Cone, whoſe Axe is AX, baſe BCD, and plane *b x d* drawn, at pleaſure, parallel to the baſe BCD. And ſince, as *Diam.* BD : *Periph.* BCD : : *Diam. bd.* : *Periph. b x d*, and ſo every where it will be (according to the *Method of Indiviſibles*, and by 12. 5.) as *Diam.* BD, to *Periph.* BCD, ſo is the triangle ABD, conſiſting of thoſe parallel Diameters, to the *Conic Superficies* ABCD, conſiſting of thoſe Peripheries, *i. e.* *Diam.* BD : *Periph.* BCD : : $\frac{AX \times BD}{2} : \frac{AX \times \textit{Periph.}\ BCD}{2}$. Whence $\frac{AX \times \textit{Periph.}\ BCD}{2}$ will be equal to the ſuperficies of the Cone; which is falſe and contrary to what was demonſtrated juſt now. For we demonſtrated that the ſuperficies of the Cone was $\frac{AB \times \textit{Periph.}\ BCD}{2}$.

I In

In anſwering this Objection, we ſay, that the *Method of Indiviſibles*, in the ſpeculation of Perimeters, and of Curve Surfaces, proceeds otherwiſe than in the ſpeculation of plane Surfaces and ſolid Contents. It does indeed ſuppoſe that the Area of plane Figures conſiſts as it were of parallel right-lines, and the contents of ſolids of parallel Planes, and that their number may be expreſs'd by the altitude of the Figures: But it by no means ſuppoſes, that the Perimeters of plane figures conſiſt of points, or the ſuperficies of ſolids of lines, the number of which may be expreſs'd by the altitude of the figure. As for example, altho' the triangle ABD (in the laſt figure) conſiſts of lines parallel to BD, the number of which is expreſſed by the number of points in the perpendicular AX, that is, by the length of the perpendicular: Yet it would be abſurd to ſuppoſe that the line AB conſiſts of points, whoſe number may be expreſs'd by the number of points in a leſs line AX. For altho' the right-lines *b d* drawn thro' each infinitely ſmall part of AX, divide AB into as many infinitely ſmall parts, yet thoſe parts are not of the ſame Denomination or Quality with the parts of AX, but ſomewhat greater than them; ſo that if the parts of AX be look'd upon as points, the parts of AB are not to be called points, but greater than points; and on the contrary, if the parts of AB be called points, the parts of AX are to be look'd upon as leſs than points, if it be lawful to ſpeak ſo. For the points which are treated of in the *Method of Indiviſibles* are not abſolutely points, but indefinitely ſmall parts, which uſurp the name of points, becauſe of their affinity to them. Since therefore points don't admit of greater and leſs, the name of points is not at the ſame time to be attributed to the parts of different magnitudes; conſequently, tho' the number of the greater parts of AB may be expreſs'd by the number of the leſſer parts of AX, yet the number of points in AB can no ways be expreſſed by the number of points in AX, (that is, by the number of parts in AX, equal to the number of parts in AB, which are called points.) The line AB has as many points as there are in it ſelf alone, or another line equal to it ſelf, nor can it be determined by any other meaſure. After the ſame manner, this method don't ſuppoſe the conic Surface ABCD to conſiſt of as many parallel circumferences perpetually increaſing from the vertex A, or decreaſing from the baſe BD, as there are points in the Axe AX, but rather of as many thus increaſing or decreaſing as there are points in the ſide AB. For in the Revolution of the line AB about the Axis AX, (whereby the ſuperficies of the Cone is generated) every point in the line AB produces a circumference, and conſequently more circumferences are produced than the points contained in the Axis AX. Therefore if you would extend the *Method of Indiviſibles* to the ſuperficies of ſolids, and ſuppoſe thoſe ſuperficies to conſiſt of parallel-lines, you ought not to compute this by the parallel Areas conſtituting the ſolids, that is, not to number thoſe Areas by the altitude of the ſolid, but by other lines agreeable to the condition of each figure. Which lines, in figures that are not irregular, may eaſily be determined: For inſtance, in the equilateral Pyramid ABCD, (*Fig.* 16.) whoſe Axe is AX, ſuppoſing that the lateral ſurface of the Pyramid conſiſts of Perimeters of triangles, parallel to the baſe BCD, theſe can neither be computed by the altitude AX, nor by the ſide AB, (for by the former, the thing required would be wanting of the true Dimenſion, and by the latter 'twould exceed it) but by the line AE drawn from the vertex A pependicular to the ſide BC of the baſe: The reaſon of which is, that every plane ſide of a Pyramid, as ABC, conſiſts of parallel right-lines computed by the altitude AE. After the ſame manner, ſuppoſing that the ſuperficies of the Hemiſphere BAD, (*Fig.* 17.) conſiſts of Peripheries of circles parallel to the baſe BCD, the number of them is not to be computed by the Axis AX, but by the Quadrantal Arc AB, becauſe that every point of the Arc AD in revolving produces a circumference; and ſo any ſuperficies, whether plane or curv'd, which is conceived to conſiſt of equidiſtant right or curv'd lines, is to be computed by a line cutting thoſe equidiſtant lines perpendicularly. For ſince thoſe equidiſtant lines, in this *Method of Indiviſibles*, are not conſidered abſolutely as lines having an infinitely ſmall breadth, which is the ſame with the breadth or thickneſs of the point deſcribing thoſe equidiſtant lines in their Circumvolution, and ſince the ſame equidiſtant lines divide the line cutting them perpendicularly into parts meaſuring its breadth, thoſe parts are to be looked upon as ſuch ſort of points, and conſequently the number of equidiſtant

diſtant lines, or the ſum of thoſe breadths is to be computed by the number of points in the line cutting them perpendicularly, that is, by the length of that line, and not by a line of any other length, for that will conſiſt of more or leſs points.

Hence therefore in the ſpeculation of the ſuperficies of ſolids, the *Method of Indiviſibles* is not unuſeful, but rather very commodious, provided it be rightly underſtood, and applied according to the Rule preſcribed. For by the help of it even theſe ſuperficies may be found, if we have ſome convenient *Data* preſuppoſed, on which the reaſoning may be founded: For inſtance, we might, by the help of it, inveſtigate the ſuperficies of a Cone, by reaſoning after this manner.

If the ſuperficies of the Cone ABC (*Fig.* 15.) be divided into innumerable Peripheries of circles *b x d* parallel to the baſe BCD, the breadth of thoſe Peripheries taken together, make up the ſide AB cutting them perpendicularly, and conſequently there will be as many Peripheries as there are points in the line AB, that is, their number may be expreſs'd by the number of points in AB, or by its length. Wherefore if you draw perpendiculars equal to the Peripheries to every point of AB, a ſuperficies will be made out of thoſe perpendiculars equal to the ſuperficies of the Cone; but that ſuperficies will be a triangle whoſe height is AB, and baſe equal to the greateſt Periphery BDC, and ſo the ſuperficies of the Cone will be $= \frac{1}{2}$ AB × *Periph.* BDC, which concluſion agrees with the things laid down and demonſtrated by *Archimedes*.

After the ſame manner, if you take any right-line *a b* (*Fig.* 18.) equal to the quadrantal Arc AB of the Hemiſphere (*Fig.* 17.) and to each of its points *m* let the perpendiculars *m n* be erected equal to the Radii MN of the parallel circles MOM paſſing through the correſponding points M of that quadrantal Arc, the greateſt of which *b x* let be equal to the Radius BX of the baſe of the Hemiſphere: The figure *a b x* will contain the Radii of all the circles of whoſe Peripheries the ſuperficies of the Hemiſphere conſiſts. And if the perpendicular *m o*, *b d* be erected equal to the Peripheries MO, BD, there will be made the figure *a b d* equal to the ſuperficies of the Hemiſphere. The dimenſion of which figure if you can by any means find (as in this caſe you are to find the Area of the figure *a b x*) thence you will eaſily deduce the content of the ſegment of the ſphere, agreeing to what you would gather by any other genuine method of reaſoning. Which Obſervation, I think, will not be unuſeful in *Geometry.*

# AN

# APPENDIX,

## CONTAINING

## The NATURE, CONSTRUCTION, and APPLICATION

## OF

# LOGARITHMS.

LOgarithms are a ſet of artificial numbers ſo proportioned among themſelves, and adapted to the natural numbers, ſo as to perform the ſame things by addition and ſubtraction, as theſe do by multiplication and diviſion.

From this definition it follows, 1. That in any ſyſtem or table of Logarithms whatever, the Logarithm of unity or 1, will be nothing: for as one neither increaſes nor diminiſhes the number multiplied by it, ſo neither will its Logarithm either increaſe or diminiſh the Logarithm to which it is added; and conſequently the Logarithm of 1 muſt be nothing.

2. For a like reaſon, the Logarithm of a proper fraction will always be negative; for ſuch a fraction always diminiſhes the number multiplied by it, and therefore its number will always diminiſh the Logarithm to which it is added.

3. This property of Logarithms, affords us no ſmall compendium in multiplication; for whenever one number is to be multiplied by another, it is but taking out their Logarithms, and adding them together, and their ſum will be a third Logarithm, whoſe natural number being taken out of the tables will be the product required.

4. The ſubtraction of Logarithms anſwers to the diviſion of natural numbers to which they belong; that is, whenever one number is to be divided by another, it is but ſubtracting the logarithm of the diviſor from the Logarithm of the dividend, and the remainder will be the Logarithm

of the quotient; and becauſe every fraction is nothing elſe but the quotient of the numerator divided by the denominator, its Logarithm will be found by ſubtracting the Logarithm of the denominator from the Logarithm of the numerator. To demonſtrate this let the number $x$ be divided by the number $y$; and let the quotient be the number $z$; and let the Logarithms of the numbers $x, y, z$, be $a, b, c$, reſpectively; I ſay then that $a - b = c$; for ſince by the ſuppoſition $\frac{x}{y} = z$, we ſhall have $x = yz$, and $a = b + c$ by the definition; whence $a - b = c$.

5. As every fourth proportional is found by multiplying the ſecond and third numbers together, and dividing the product by the firſt, ſo the Logarithm of every ſuch fourth proportional will be found by adding the Logarithms of the ſecond and third numbers together, and ſubtracting from the ſum the Logarithm of the firſt This renders all the operations by the rule of proportion very compendious and eaſy; this compendium is chiefly uſeful in Trigonometry, both plain and ſpherical, which it renders very eaſy to be performed.

6. If $x$ be any number, whoſe Logarithm is $a$, then the Logarithm $x^2$ will be $2a$, that of $x^3$, $3a$, &c. that of $\frac{1}{x}$ $-a$, that of $\frac{1}{x^2}$, $-2a$, &c. And univerſally, the Logarithm of $x^m$ will be $a \times m$, and that, whether the index $m$ be intregal or fractional, affirmative or negative: on the other hand if $q$ be the Logarithm of any power of $x$, as of $x^m$, then $\frac{q}{m}$ will be the Logarithm of $x$. The reaſon of all this is plain; for as $x^2$ is the product of $x$ multiplied into itſelf, ſo its Logarithm will be the Logarithm of $x$ added to itſelf or doubled, that is $2a$; and ſo of the higher powers. Again, as $\frac{1}{x}$ is the quotient of unity divided by $x$, its Logarithm will be found by ſubtracting $a$ from $0$, the Logarithm of 1, which gives $-a$; and ſo of the other powers. Laſtly, as $\sqrt{x}$, when multiplied into itſelf produces $x$, ſo its Logarithm when added to itſelf ought to make $a$; therefore the Logarithm of $\sqrt{x}$ will be $\frac{1}{2}a$, and ſo of all the other fractional powers.

7. If any ſet of numbers $1, x, xx, x^3$, &c. be in continual geometrical progreſſion, their Logarithms, $0, a, b, c$, &c,

&c. will be in arithmetical progreſſion: For ſince by the ſuppoſition 1 is to $x$, as $x$ is to $xx$, as $xx$ is to $x^3$; that is, ſince $\frac{x}{1} = \frac{xx}{x} = \frac{x^3}{xx}$, we ſhall have $a - o = b - a = c - b$ by the fourth conſectory; therefore $o$, $a$, $b$, $c$, are in arithmetical progreſſion.

In the progreſſional ſeries of geometrical proportionals, if between the terms 1 and $x$ be inſerted a mean proportional, which is $\sqrt{x}$, its index or Logarithm will be $\frac{1}{2}$, becauſe its diſtance from unity will be but one half of the diſtance of $x$ from unity, and conſequently the root of $x$ will be expreſſed by $\frac{1}{2}$. In like manner, if between $x$ and $x^2$ we inſert a mean proportional, its index will be 1 and $\frac{1}{2}$ or $\frac{3}{2}$, becauſe its diſtance from unity is once and half the diſtance of $x$ from the ſame place of unity.

Again, if between 1 and $x$ be inſerted two mean proportionals; the firſt of theſe will be the cube root of $x$ or $\sqrt[3]{x}$, and its index will be $\frac{1}{3}$, becauſe its diſtance from the units place is but one third of the diſtance of $x$ from the ſame place of unity; and conſequently $\sqrt[3]{x}$ is expreſſed by $x^{\frac{1}{3}}$; whence it follows, that the index, or Logorithm of unity is nothing, as we have before obſerved; becauſe unity cannot be removed at any diſtance from itſelf.

And the ſame ſeries of geometrical proportionals may be continued on the contrary ſide of unity, or towards the left-hand, and which will therefore decreaſe in the ſame proportion as the terms placed on the right-hand increaſes.

For the terms $\frac{1}{x^5}$, $\frac{1}{x^4}$, $\frac{1}{x^3}$, $\frac{1}{x^2}$, $\frac{1}{x}$, 1, $x$, $x^2$, $x^3$, $x^4$, $x^5$, &c. are in the ſame geometrical progreſſion continued, and have all the ſame common ratio; therefore becauſe the diſtance of $x$ from unity, towards the right-hand is poſitive, or $+1$, the diſtance of $\frac{1}{x}$ from unity towards the left-hand, which is equal to the former, is negative, or $-1$, conſequently the index of $\frac{1}{x}$ may be written $x^{-1}$.

Alſo, becauſe $x^2$ is on the right ſide of unity, and its index $+2$, the index of $\frac{1}{x^2}$, which is as far diſtant from unity on the contrary ſide, or towards the left-hand, will be $-2$, and conſequently $\frac{1}{x^2}$ is the ſame with $x^{-2}$. In like manner $x^{-3}$ is the ſame with $\frac{1}{x^3}$, and $x^{-4}$ the ſame with $\frac{1}{x^4}$; whence it follows that theſe negative indices ſhew that the terms to which they belong, are as far diſtant from unity towards the left-hand, as the terms whoſe indices are the ſame and poſitive, are removed from unity on the contrary ſide, or towards the right-hand.

Again, if between 1 and $x$, in the ſeries increaſing from unity there be inſerted a mean proportional as $\sqrt{x}$, and between this mean proportional, and unity be inſerted another mean proportional, as $\sqrt[4]{x}$, and between this laſt mean proportional and unity be inſerted another mean proportional as $\sqrt[8]{x}$, &c. thoſe reſpective indices will be $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. likewiſe if between unity and the next term $\frac{1}{x}$ decreaſing be inſerted a mean proportional, as $\frac{1}{\sqrt{x}}$, the correſpondent index will be $-\frac{1}{2}$; and if between this mean proportional and unity be inſerted another mean proportional as $\frac{1}{\sqrt[4]{a}}$ the correſpondent index will be $-\frac{1}{4}$, &c. And as the ſame may be done between any other two terms of the progreſſion, it follows, that between any two terms may be inſerted an infinite number of mean proportionals, whoſe reſpective indices will become the Logarithms of the reſpective terms to which they belong: And theſe mean proportionals the greater they are in number, between the two terms of any ratio, the nearer they approach to a ratio of equality, with their correſpondent indices or exponents.

Hence we ſee the reaſon of the ancient method of making Logarithms, and which was practiced by their firſt inventors, which was, to extract the ſquare root, out of the ſquare root, &c. of any number, in order to find a ſeries of continual mean geometrical proportionals; till the number of cyphers contained bewteen unity, and the firſt ſignificant figure of the root, was equal to the number of places the intended Logarithm ſhould conſiſt of; and at the ſame time to find out a like number of correſpondent arithmetical means, which would be the Logarithms of the geometrical means reſpectively.

But how tedious and opperoſe this method is, may be eaſily imagined, by any one who has been at the pains to extract the root of a large number; and may be gathered from this, that to find a Logarithm to ſeven places only, requires at leaſt 27 extractions of root out of root, the ſquare conſiſting of 16 figures at leaſt; and if in any one of the operations an error happens, the whole work muſt be repeated.

From what has been ſaid we may obſerve that the Logarithm of any number, is the Logarithm of the ratio of unity, to the number itſelf; or it is the diſtance between unity and the ſame number, in the geometrical ſcale of proportionals, and is meaſured by the number of proportionals contained between them: Logarithms therefore expound the place or order that every number obtains from the units place in the continued ſeries or ſcale of proportionals, infinite in number.

Thus, if between unity and the number 10, there be ſuppoſed 10,000,000, *&c.* mean proportionals, that is, if the number 10 be placed in the 10,000,000th the place from unity, between 1 and 2 there will be found 3010300 of ſuch proportionals; that is, the number 2 will be placed in the 3010300th place from unity; between 1 and 3, there will be found 4771213 of the ſame proportionals, or the number 3 will ſtand in the 4771213th place, in the infinite ſcale of proportionals, which numbers 10000000, 3010300, 4771213, are therefore the Logarithms of 10, 2, and 3; or, more properly, the Logarithms of the ratio's of thoſe numbers one to the other.

Again, if between unity and the number 10 there be ſuppoſed an infinite ſcale of mean proportionals, whoſe number is 23025851, *&c.* that is, if the number 10 be placed in the 23025851ſt place from unity, in the infinite ſcale of proportionals between 1 and 2 there will be found to be 6931471, *&c.* ſuch proportionals; and between 1 and three 10986122, *&c.* of the ſame proportionals; ſo that if the firſt term of the ſeries be called $x$, the ſecond term will be $x^2$, the third $x^3$, *&c.* and if the number 10 be ſuppoſed to be the 10000000*th* term in the ſeries, the number 2 will be the 3010300*th* term, and the number 3 will be the 4771213*th* term in the ſame ſeries; and conſequently $x^{10000000} = 10$, $x^{3010300} = 2$, and $x^{4771123} = 3$, *&c.* and again, if the number 10 be ſuppoſed the 23025851*ſt* term, in the ſeries, the number 2 will be the 6931471ſt term, and the number 3, the 10986122*nd* term in the ſame ſeries; and conſequently, in this caſe, $x^{23025851} = 10$, $x^{6931471} = 2$, and $x^{10986122} = 3$, *&c.*

And as the infinite number of mean proportionals between any two numbers may be aſſumed at pleaſure, hence Logarithms may be of as many different forms (that is, there may be as many different ſcales of Logarithms) as there can be aſſumed different ſcales of mean proportionals between any two numbers. Every number therefore is ſome certain power of that number which is placed next to unity; in the infinite ſcale of proportionals between unity and the given number, and the index of that power is the Logarithm of the number.

Logarithms therefore may be of as many ſorts as you can aſſume different indices of the power of that number whoſe Logarithm is required,

And becauſe the ratio of unity to any number is meaſured by the number of ratiuncula contained in that ratio; we may therefore value ratio's by the number of ratiuncula contained in each ratio, and may conſider them as *Quantates ſui generis*, beginning from the ratio of 1 to 1 = 0, being affirmative when the root is increaſing, but negative when decreaſing.

So that ratio's are to each other as the number of like and equal ratiuncula contained between the two terms of the ratio; wherefore, if the ratio of 1 to 10 contain 10000000, *&c.* equal ratiuncula, that of its duplicate, or of 10 to 100, will contain 20000000, *&c.* that of its triplicate, or of 100 to 1000, 30000000, *&c.* of the ſame ſimilar and equal ratiuncula; ſo that the Logarithms of numbers, or the values of ratio's, are in an arithmetical progreſſion; and hence ariſeth the common definition of Logarithms, *viz.* that Logarithms are a ſeries of numbers in an arithmetical progreſſion fitted or aſſigned to a rank of numbers in a geometrical progreſſion.

If the diſtance between unity and any number be ſuppoſed to conſiſt always of the ſame number of mean proportionals as 10000000, *&c.* or, which is the ſame thing, if we make the ratio of unity to any number, to conſiſt always of the ſame infinite number of ratiuncula, the magnitudes of each in this caſe, will be, as their number in the former; ſo that if between unity and any number propoſed, there be taken an infinity of mean proportionals, the infinitely ſmall augment or decrement of the firſt of thoſe means from unity, will be a ratiuncula; and becauſe this is always proportional to the momentum or fluxion of the ratio of unity to the ſaid nummber; and ſince in theſe continual proportionals all the ratiuncula are equal, their ſum, or the whole ratio, will be as the ſaid momentum or fluxion is directly; wherefore if the root of the infinite powers be extracted out of any number, the difference between that root and unity, will be as the Logarithm of that number; ſo that the infinite root of any number being found, the Logarithm of that number is eaſily computed. How the infinite root of any number may be found, muſt be next conſidered.

Let $p + px$ be a given quantity whoſe $n$ root is to be extracted; where $n$ ſtands for any power or number taken at pleaſure.

Then will $\overline{p+px}^{\,n} = p^{n} \times \overline{1+x}^{\,n}$.

Aſſume $\overline{1+x}^{\,n} = 1 + Ax + Ba^2 + Cx^3 + Dx^4 + Ex^5$, &c.

Then $n \times \overline{1+x}^{\,n-1}\,\dot{x} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x} + 4Dx^3\dot{x} + 5Ex^4\dot{x}$, &c.

And $n \times \overline{1+x}^{\,n-1} = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4$, &c.

And $\dfrac{n \times \overline{1+x}^{\,n-1}}{\overline{1+x}^{\,n}} = \dfrac{A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4, \text{ \&c.}}{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5, \text{ \&c.}}$

And $n + nAx + nBx^2 + nCx^3 + nDx^4$, &c. $= A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4$, &c.

Alſo $n = A$, and $nA = 2B + A$.
And $nA - A = 2B$.
And $\overline{n-1} \times A = 2B$.
And $\dfrac{n-1}{2} \times A = B$.

Alſo $nB = 3C + 2B$.
And $nB - 2B = 3C$.
And $\overline{n-2} \times B = 3C$.
And $\dfrac{n-2}{3} \times B = C$.

Alſo $nC = 4D + 3C$,
And $nC - 3C = 4D$,
And $\overline{n-3} \times C = 4D$,
And $\dfrac{n-3}{4} \times C = D$.

Alſo $nD = 5E + 4D$.
And $nD - 4D = 5E$.
And $\overline{n-4} \times D = 5E$.
And $\dfrac{n-4}{5} \times D = E$.

Wherefore $A = n$, $B = n \times \dfrac{n-1}{2}$, $C = n \times \dfrac{n-1}{2} \times \dfrac{n-2}{3}$, $D = n \times \dfrac{n-1}{2} \times \dfrac{n-2}{3} \times \dfrac{n-3}{4}$, and $E = n \times \dfrac{n-1}{2} \times \dfrac{n-2}{3} \times \dfrac{n-3}{4} \times \dfrac{n-4}{5}$, &c.

And, $\overline{1+x}^{\,n}=1+nx+n\times\frac{n-1}{2}x^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}x^3+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}x^4$, &c.

And $\overline{p+px}^{\,n}=p^n+n\,p^n\,x+n\times\frac{n-1}{2}p^n\,x^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}p^n\,x^3$, &c.

Put $p^n=A$, $np^nx=B$, $n\times\frac{n-1}{2}p^n\,x^2=C$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}p^n\,x^3=C$.

Then $\overline{p+px}^{\,n}=p^n+nAx+\frac{n-1}{2}Bx+\frac{n-2}{3}Cx+\frac{n-4}{5}Ex$, &c. Which is the celebrated binomial ſeries invented by the illuſtrious Sir *Iſaac Newton*, for raiſing a given quantity to any given power, or extracting any root out of the ſame quantity; becauſe $n$ ſtands for any number whether it be a whole number or a fraction. When $n$ is finite, $\overline{1+x}^{\,\frac{1}{n}}=1+\frac{1}{n}x+\frac{1-n}{2nn}x^2+\frac{1-3n+2n^2}{6n^3}x^3+\frac{1-6n+11n^2-6n^3}{24n^4}x^4$, &c. but when it is infinite $\overline{1+x}^{\,\frac{1}{n}}=1+\frac{1}{n}x-\frac{1}{2n}x^2+\frac{1}{3n}x^3-\frac{1}{4n}x^4+\frac{1}{5n}x^5$, &c. $nn$ being infinitely infinite, and conſequently whatever is divided by it vaniſhes; whence it follows, that $\frac{1}{n}$ multiplied into $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$, &c. is the augment of the firſt of the mean proportionals between unity and $1+x$, and is therefore the Logarithm of the ratio of 1 to $1+x$.

But becauſe the infinite root of $1-x$, or $\overline{1-x}^{\,\frac{1}{n}}$ is $1-\frac{1}{n}x\ \frac{1}{3n}x\ 1-\frac{1}{n}x-\ \frac{1}{2n}x^2\ \frac{1}{3n}x^3-\frac{1}{4x}x^4$, &c. therefore $\frac{1}{n}\times$ into $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5$, &c. will be the Logarithm of the ratio of 1 to $1-x$, or the Logarithm of a number leſs than unity; and whereas the infinite index $n$ may be taken at pleaſure, the ſeveral ſcales of Logarithms to ſuch indices will be reciprocally as $\frac{1}{n}$, or as the ſeveral indices, and if the index be taken = 10000000, &c. the reſpective Logarithms will be ſimply,

$$x\pm\frac{1}{2n}x^2+\frac{1}{3n}x^3\pm\frac{1}{4n}x^4+\frac{1}{5n}x^5,\ \text{\&c.}$$

Let $a$ repreſent the leaſt of any two given numbers, $b$ the greateſt, $z$ the ſum of the two numbers, and $d$ the difference, and let us ſuppoſe the ratio of $a$ to $b$, to be divided into that of $a$ to $\frac{1}{2}z$, and of $\frac{1}{2}z$ to $b$; that is into the ratio of $a$ to the arithmetical mean between the two numbers, and the ratio of the ſaid arithmetical mean to the greater number $b$; then becauſe $\frac{a}{\frac{1}{2}z}\times\frac{\frac{1}{2}z}{b}=\frac{a}{b}$, the Logarithm of $\frac{a}{\frac{1}{2}z}$ + Log. $\frac{\frac{1}{2}z}{b}$ = Log. of $\frac{a}{b}$, or becauſe $\frac{\frac{1}{2}z}{a}\times\frac{b}{\frac{1}{2}z}=\frac{b}{a}$ the Log. of $\frac{\frac{1}{2}z}{a}$ + Log. $\frac{b}{\frac{1}{2}z}$ = Log. $\frac{b}{a}$; that is, the ſum of the Logarithms of theſe two ratio's will be the Log. of the ratio of $a$ to $b$; and to find the value of each of theſe ratio's in the terms of 1 and $1\pm x$ we muſt ſay, as $\frac{1}{2}z : a :: 1 : 1-x$, whence $x=\frac{b-\frac{1}{2}z}{\frac{1}{2}z}=\frac{d}{z}$, and ſubſtituting $\frac{d}{z}$ in the room of $x$, in each of the former ſeries, we ſhall have $\frac{1}{n}\times\frac{d}{z}+\frac{d^2}{2z^2}+\frac{d^3}{3z^3}+\frac{d^4}{4z^4}+\frac{d^5}{5x^5}$, &c. for the Log. of the ratio of $a$ to $\frac{1}{2}z$, and $\frac{1}{n}\times\frac{d}{z}-\frac{d^2}{2z^2}+\frac{d^3}{3z^3}-\frac{d^4}{4z^4}+\frac{d^5}{5z^5}$, &c. for the Log. of the ratio $\frac{1}{2}z$ to $b$,; and conſequently the ſum of theſe two ſeries $\frac{1}{n}\times\frac{2d}{z}+\frac{2d^3}{3z^3}+\frac{2d^5}{5z^5}$, &c. or $\frac{1}{n}\times 2\times\frac{d}{2}+\frac{d^3}{3z^3}+\frac{d^5}{5z^5}+\frac{d^7}{7z^7}+\frac{d^9}{9z^9}$, &c. will be the Logarithm of the ratio of $a$ to $b$, whoſe difference is $d$, and ſum $z$; whence to find the Logarithm of any prime number we have this general rule.

"To the given number add 1 for a denominator or diviſor, and ſubtract 1 from the ſame number for a numerator or dividend, then of the vulgar fraction thence reſulting, compoſe all the odd powers thereof, theſe will form a ſeries of numbers which being divided by their reſpective exponents, *viz.* 1, 3, 5, 7, 9, *&c.* will produce a ſeries of quotes, whoſe ſum will be the natural Logarithm of the number propoſed, and being doubled, will give the Logarithm of the ſame number, according to *Neper*'s form."

This rule is not only very eaſy to be retained in memory, but very proper for the practice of making Logarithms, which it performs with the greateſt expedition, and is ſo very plain that it may be taught to perſons of a mean capacity.

For example, Let it be required to find the Logarithm of 2, the firſt prime number.

Becauſe $a = 1$, and $b = 2$, therefore $d = b - a = 1$, and $z = a + b = 3$; wherefore $\frac{d}{z} = \frac{1}{3}$, and $\frac{d^2}{z^2} = \frac{1}{9}$.

Hence $\frac{1}{3} + \frac{1}{3} \times \frac{1}{27} + \frac{1}{5} \times \frac{1}{243} + \frac{1}{7} \times \frac{1}{2187}$, *&c.* $= \frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15309}$, will be the natural Logarithm of 2, and conſequently $\frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \frac{2}{15309}$, *&c.* will be the Logarithm of the ſame number in *Neper*'s form; wherefore if the ſeveral fractions be added together, and reduced into an equivalent decimal, we ſhall have the Logarithm of the given number; but becauſe to multiply by $\frac{1}{9}$ and divide by 9 is the ſame thing, if the firſt fraction be reduced into a decimal, and that divided by 9, and each ſucceſſive quotient by the ſame, we ſhall have a ſeries of quotients, which being divided by the ſeveral co-efficients 1, 3, 5, 7, 9, *&c.* the ſum of theſe laſt quotients will be the Logarithm of the number given; and in order to obtain the Logarithm true to any number of places, it will be neceſſary to continue the fraction to one or two more places, than the intended number of places the Logarithm is to conſiſt of; ſo that to have the Logarithm true to 10 places, it will be convenient to find the value of $\frac{2}{3}$ in decimals to 11 or 12 places, as in the following operation.

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{3}$ = | 333. 333. 333. 333 | | | = 333. 333. 333. 333 | + |
| $\frac{1}{9}$ = | 37. 037. 037. 037 | its | $\frac{1}{3}$ | = 12. 345. 679. 012 | + |
| $\frac{1}{9}$ = | 4. 115. 226. 337 | its | $\frac{1}{5}$ | = 823. 045. 268 | — |
| $\frac{1}{9}$ = | 457. 247. 371 | its | $\frac{1}{7}$ | = 65. 321. 053 | |
| $\frac{1}{9}$ = | 50. 805. 263 | its | $\frac{1}{9}$ | = 5. 645. 029 | + |
| $\frac{1}{9}$ = | 5. 645. 029 | its | $\frac{1}{11}$ | = 513. 185 | — |
| $\frac{1}{9}$ = | 627. 225 | its | $\frac{1}{13}$ | = 48. 248 | |
| $\frac{1}{9}$ = | 69. 691 | its | $\frac{1}{15}$ | = 4. 646 | |
| $\frac{1}{9}$ = | 7. 743 | its | $\frac{1}{17}$ | = 456 | — |
| $\frac{1}{9}$ = | 860 | its | $\frac{1}{19}$ | = 45 | + |
| $\frac{1}{9}$ = | 96 | its | $\frac{1}{21}$ | = 5 | — |

Whence the natural Log. of 2 is 346. 573. 590. 280

2

And the *Neper*'s Log. of 2 is——693. 147. 180. 560

Hence if the *Neper*'s Log. of 2 be doubled will give 1386. 294. 361. 120, for the *Neper*'s Log. of 4, and this again being doubled will give 2772. 588. 722. 240, for the *Neper*'s Log. of 16, *&c.* Or if the Log. of 2 be trebled, or multiplied by 3, we ſhall have 2079. 441. 541. 680, for the Log. of 8, *&c.*

And, if the Logarithm of 8 be multiplied by the Log. of $1\frac{1}{4}$, we ſhall have the Log. of 10; and to find the Log. $1\frac{1}{4}$ we have $a = 1$, $b = 1\frac{1}{4} = \frac{5}{4}$, whence $z = \frac{9}{4}$, and $d = \frac{1}{4}$, and conſequently $\frac{d}{z} = \frac{1}{9}$, and $\frac{d^2}{z^2} = \frac{1}{81}$. From whence the Log. of 10 will be found as follows.

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{9}$ = | 111. 111. 111. 111 | | | = 111. 111. 111. 111. |
| $\frac{1}{81}$ = | 1. 371. 742. 112 | its | $\frac{1}{3}$ | = 457. 247. 371 |
| $\frac{1}{81}$ = | 16. 935. 087 | its | $\frac{1}{5}$ | = 3. 387. 017 |
| $\frac{1}{81}$ = | 209. 075 | its | $\frac{1}{7}$ | = 29. 868 |
| $\frac{1}{81}$ = | 2. 851 | its | $\frac{1}{9}$ | = 287 |
| $\frac{1}{81}$ = | 32 | its | $\frac{1}{11}$ | = 3 |

The natural Log. of $1\frac{1}{4}$ is —— 111. 571. 775. 657

2

And the *Neper*'s Log. of $1\frac{1}{4}$ is—— 223. 143. 551. 314

The Log. laſt found being added to the Log. of 8 found before, we ſhall have 2302. 585. 092. 994, for the *Neper*'s Log. of 10.

But as neither the natural Logarithms, nor thoſe of *Neper*, the celebrated inventor of Logarithms, are ſo proper for calculation as could be wiſhed, the Lord *Neper* himſelf, with the aſſiſtance of Mr. *Henry Briggs*, altered their form, and made the Logarithm of 10 to be 1,0000000000, *&c.*

and not 2.30258509299, *&c.* that is, he made the number 10 the 10000000000th term in the ſeries, whence the Logarithm of 100 will be 2000000000, *&c.* and the Logarithm of 1000, will be 30000000000, *&c.* whence the Logarithms between 1 and 10 will begin from 0, that is, they will have 0 for the firſt term towards the left-hand, becauſe they are each of them leſs than the Logarithm of 10, which has unity or 1 in the firſt place; in like manner the Logarithms of all numbers betwixt 100 and 1000 begin from 2, becauſe they are greater than the Logarithm of 100, which is fixed at 2, and leſs than the Logarithm of 1000, which is made equal to 3, *&c.*

The firſt figure of every Logarithm towards the left-hand, which is ſeperated from the reſt by a point, is called the index of that Logarithm; becauſe it points out the higheſt or remoteſt place of that number from the place of unity in the infinite ſcale of proportionals towards the left-hand. Thus if the index of the Logarithm be one, it ſhews that its higheſt place towards the left-hand is the tenth place from unity; and therefore all Logarithms which have 1 for their index will be found between the tenth and hundredth place in the order of numbers. And for the ſame reaſon all Logarithms which have 2 for their index will be found between the hundredth and thouſandth place in the order of numbers, *&c.* whence the index or characteriſtic of any Logarithm is always leſs by one than the number of figures in whole numbers which anſwer to the given Logarithm.

As all ſyſtems of Logarithms whatever are compoſed of ſimilar quantities, it will be eaſy to form, from any ſyſtem of Logarithms another ſyſtem in any given ratio, and conſequently to reduce one table of Logarithms into another of any given form. As for inſtance, in the ſyſtem given let $a$, $b$, and $c$ be the Logarithms of the three given numbers A, B, and C, whereof the third is equal to the product of the other two multiplied together. Then will $a+b=c$ by the definition we have given of Logarithms. Let us now imagine all the Logarithms of this given ſyſtem to be doubled, then will $a$, $b$, and $c$ be changed into $2a$, $2b$, $2c$; but as $a+b$ was equal to $c$ in the former ſyſtem, ſo in the latter $2a+2b$ will be equal to $2c$. Whence it follows that we may reduce one ſyſtem of Logarithms into another of any given form by ſaying, as any one Logarithm in the given form is to its correſpondent Logarithm in the new form; ſo is any other Logarithm in the given form, to its correſpondent Logarithm in the required form; and hence we are taught how to reduce either *Neper*'s or the natural Logarithms into the form of *Brigg*'s, and the contrary.

For as 2. 302. 585. 092. 994, the *Neper*'s Logarithm of 10, to 1.0000000000, the *Brigg*'s Logarithm of 10; ſo is any other Logarithm in *Neper*'s form, to the correſpondent tabular Logarithm in *Brigg*'s form: And becauſe the two firſt numbers conſtantly remain the ſame; if the *Neper*'s Logarithm of any one number be divided by 2.302.585, *&c.* or multiplied by 434. 294. 481. 903, the ratio of 1.00000, *&c.* to 2. 302. 58, *&c.* as is found by dividing 1.000000, *&c.* by 2. 302. 58, *&c.* the quotient in the former, and the product in the latter will give the correſpondent Logarithm in *Brigg*'s form, and the contrary. And after the ſame manner the ratio of natural Logarithms to that of *Brigg's* will be found = 868. 588. 963. 806.

If inſtead of finding a decimal fraction equal to $\frac{d}{z}$ we divide the reciprocal ratio of the *Neper*'s or the natural Logarithms, to that of *Brigg*'s by the reciprocal ratio of $d$ to $z$, and divide that quotient by the ſquare of $z$, we ſhall have a ſeries of quotients, which being divided reſpectively by the ſeveral co-efficients 1, 3, 5, 7, 9, *&c.* will produce a new ſeries of quotients, whoſe ſum will give the *Brigg*'s Logarithm of the number propoſed.

Let it therefore be required to find the *Brigg*'s Logarithm of 2 to ten places, from the reciprocal ratio of the natural to the *Brigg*'s Logarithm.

Put R = 868. 588. 963. 806, and becauſe $a=1$, $b=2$, $d=1$, and $z=3$; therefore $R \times \overline{\frac{1}{3} + \frac{1}{3} \times \frac{1}{27} + \frac{1}{5} \times \frac{1}{243} + \frac{1}{7} \times \frac{1}{2187}}$, *&c.* = the Logarithm of 2.

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{3}$ R = | 289. 529. 654. 60 | | | = | 289. 529. 654. 60 |
| $\frac{1}{9}$ —— = | 32. 169. 961. 62 | its | $\frac{1}{3}$ | = | 10. 723. 320, 54 |
| $\frac{1}{9}$ —— | 3. 574. 440. 18 | its | $\frac{1}{5}$ | = | ——714. 888. 04 |
| $\frac{1}{9}$ —— | 397. 160, 02 | its | $\frac{1}{7}$ | = | —— 56. 737. 14 |
| $\frac{1}{9}$ —— | 44. 128, 89 | its | $\frac{1}{9}$ | = | ——4. 903. 21 |
| $\frac{1}{9}$ —— | 4. 903. 21 | its | $\frac{1}{11}$ | = | —— 445. 75 |
| $\frac{1}{9}$ —— | 544. 80 | its | $\frac{1}{13}$ | = | —— 41. 91 |
| $\frac{1}{9}$ —— | 60, 53 | its | $\frac{1}{15}$ | = | ——4. 03 |
| $\frac{1}{9}$ —— | 6. 72 | its | $\frac{1}{17}$ | = | —— 40 |
| $\frac{1}{9}$ —— | 74 | its | $\frac{1}{19}$. | = | —— 4 |

The Log. of 2 in *Brigg*'s form is — 301. 029. 995. 66

Again, let it be required to find *Brigg*'s Logarithm of 3. Becauſe $a=1$, $b=3$, $d=2$, and $z=4$; therefore $\frac{d}{z} = \frac{1}{2}$.

And $\frac{R}{2}+\frac{1}{3}\times\frac{R}{8}+\frac{1}{5}\times\frac{R}{32}$, *&c.* will be the Logarithm of 3; wherefore, if R be divided by 4, and each quotient ſucceſſively by the ſame number, and thoſe again by the indices of the odd powers, the ſum of theſe laſt quotients will be the *Brigg*'s Logarithm of 3; but becauſe this ſeries converges very ſlow, the ſame Logarithm of 3 may be found much quicker, by finding the difference between the Logarithm of 2 already known, and the Logarithm of 3 required.

Put therefore $a=2$, and $b=3$, then will $z=5$, and $d=\frac{1}{5}$, and $\frac{d^2}{z^2}$, the common multiplicator equal to $\frac{1}{25}$, wherefore if R be divided by 5, and that quotient by 25, *&c.* and theſe ſeveral quotes by 1, 3, 5, 7, *&c.* the ſum of theſe laſt quotients added to the Logarithm of 2, will give the Logarithm of 3.

Operation,

$\frac{1}{5}$ R = 173. 717. 792. 76 = 173. 717. 792. 76
$\frac{1}{25}$ —— 6. 948. 711. 71 its $\frac{1}{3}$ = —— 2. 316. 237. 24
$\frac{1}{25}$ —— 277. 948. 47 its $\frac{1}{5}$ = —— 55. 589 69
$\frac{1}{25}$ —— 11. 117. 94 its $\frac{1}{7}$ = —— 1. 588. 28
$\frac{1}{25}$ —— 444. 73 its $\frac{1}{9}$ = —— 49. 41
$\frac{1}{25}$ —— 17. 79 its $\frac{1}{11}$ = —— 1. 62
$\frac{1}{25}$ —— 71 its $\frac{1}{13}$ —— 5
Diff. between the Log. of 2 and 3 = 176. 091. 259. 05
Wherefore if to the Logarithm of 2 = 301. 029. 955. 66

the difference juſt found, be added, we ſhall have the Logarithm of 3 = 477. 123. 254. 71.

Again let it be required to find the Log. of 7.

Put $a=6$, $b=7$, then will $z=13$, and $d=1$, whence $\frac{d^2}{z}=\frac{1}{169}$, and therefore $\frac{z}{d}\times R=$ 66. 814. 535. 68, conſequently.

66. 814. 535. 68 = 66. 814. 535. 68
$\frac{1}{169}$ —— 395. 352. 28 its $\frac{1}{3}$ = —— 131. 784. 09
$\frac{1}{169}$ —— 2. 339. 36 its $\frac{1}{5}$ = —— 476. 87
$\frac{1}{169}$ —— 13. 84 its $\frac{1}{7}$ = —— 1. 98
$\frac{1}{169}$ —— 8 its $\frac{1}{9}$ = —— 1 —
The diff. of the Log. 6 and 7 = 66. 946. 789. 63

Wherefore

To the Log. of 6 —— = 778. 151. 250. 38
Add the diff. found —— = 66. 946. 789. 63
The ſum is the Log. of 7 —— = 845. 098. 040. 01

The Logarithm of all compoſite numbers, that is, all ſuch as ariſe from the multiplication of others called factors, are eaſily found by the addition of the Logarithms of thoſe factors ; Thus the Log. of 4 = Log. 2 + Log. 2, and the Log. 10 = Log. 2 + Log. 5, and therefore the Log. of 5 = Log. 10 — Log. 2 ; the Log. 6 = Log. 2 + Log. 3 ; Log. 8 = Log. 2 + Log. 4 ; and the Log. 9 = Log. 3 + Log. 3. Therefore from the Logarithms of the prime numbers 2, 3, and 7, which we have already found, all the Logarithms of the numbers under 11, may be computed by addition and ſubtraction only. And after the ſame manner may the whole cannon be conſtructed, the prime numbers being eaſily computed by the Theorems we have laid down.

The excellence of this method will be the more conſpicuous if we compare the foregoing computations of the prime numbers with each other ; for then it will appear, that as the prime number itſelf increaſes, the operation decreaſes, and the fewer ſteps are requiſite ; till at laſt the firſt ſtep will be ſufficient to find the difference to any number of places requiſite.

Thus in the common tables, which extend but to ſeven places in decimals, when the difference *d* becomes equal to the one two hundredth part of the ſum *z*, the firſt ſtep, $\frac{d}{z} \times R$, will give the difference between the Logarithms true to the ſame number of places.

For if 868. 588. 9 be divided by 201, the ſum of the numbers 100 and 101, the quotient 0043213, will give the difference between the Log. of 100 and 101, true to 7 places, which therefore being added to 2.0000000, will give 2.0043213, for the Logarithm of 101 ; ſo that the Logarithms of the prime numbers under 100 being found, the Logarithms of all the prime numbers above 100 may be found by one ſingle diviſion, and the Logarithms of any intermediate numbers by addition and ſubtraction only.

As the Logarithms 1, 10, 100, 1000, 10000, &c. in *Brigg's* form are 0, 1, 2, 3, 4, &c. it is eaſy to conceive that the Logarithms of all numbers which increaſe or decreaſe in a tenfold proportion, differ from each other only in their indices or charactereſtics, the fractional number remaining conſtantly the ſame : And hence it is that the fractional number .30102999566, which is the Logarithm of 2, when the index is a cypher, becomes the Logarithm of 20 when the index is 1, of 200 when it is 2, of 2000, when it is 3, &c. Alſo the ſame Logarithm when its index is —1 becomes the Logarithm of $\frac{2}{10}$, when the index is —2 of $\frac{2}{100}$, when —3 of $\frac{2}{1000}$, &c. the fractional number ſtill remaining the ſame; for becauſe as 1 : 2 : : 10 : 20 : : 100 : 200 : : 1000 : 2000, &c. the diſtance between 1 and 2, 10 and 20, 100 and 1000, 1000 and 2000, &c. are equal to each other, and the number of terms in the infinite ſcale of proportionals between the numbers 1 and 2, 10 and 20, 100 and 200, 1000 and 2000, that is, the Logarithm of that diſtance will be the ſame; and ſince the Logarithm of 1 is 0000000000 of 10 is 1.0000000000, of 100, 2.0000000000, and the Logarithm of 2 is .30102999566, the Logarithm of 20 will be 1.30102999566, of 200, 2.30102999566, &c.

And becauſe the numbers 37251, 3725.1, 372.51, 37.251, 3.7251, .37251, .037251, .0037251, &c. are continual proportionals in the ratio of 10 to 1, the number of terms between each in the infinite ſcale of proportionals will be equal: and therefore ſince the Logarithm of 371251, is 4. 5711379358, the other numbers will be 3.5711379358, 2.5711379358, 1.5711379358, 0.5711379358, — 1.5711379358, —2.5711379358, —3.5711379358, &c.

This property of having the ſame number for the Logarithms of an infinite ſeries of numbers by increaſing or diminiſhng the characteriſtic by unity is peculial to *Brigg*'s Logarithms, which renders them preferable to any other yet invented.

It has been already ſhewn, that $\frac{1}{n} \times x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4$, &c. equal to $\overline{1+x}^{\frac{1}{n}} - 1$ will be the Logarithm of the ratio of 1 to $1+x$, where $n$ repreſents any infinite index, and $x$ any number taken at pleaſure. Now if L be put for the Logarithm itſelf, then will $\overline{1+x}^{\frac{1}{n}} - 1 = \text{L}$, conſequently, $\overline{1+x}^{\frac{1}{2}}$ will be equal to $\text{L}+1$, and $1+x = \overline{\text{L}+1}^{n}$; and becauſe $\overline{\text{L}+1}^{n} = 1 + n\text{L} + \frac{nn-n}{2}\text{L}^2 + \frac{n^3+3n^2+2n}{6}\text{L}^3 + \frac{n^4-6n^3+11n^2-6n}{24}\text{L}^4$, &c. (by the general

Theorem in page 360) when $n$ is finite, when $n$ becomes infinte $\overline{L+1}^{\,n}$ will be equal to $1 + nL + \frac{1}{2}n^2 L^2 + \frac{1}{6} n^3 L^3 + \frac{1}{24} n^4 L^4 + \frac{1}{120} n^5 L^5 + \frac{1}{720} n^6 L^5 + \frac{1}{5040} n^7 L^7$, &c. $= 1 + x$. Alſo becauſe $\frac{1}{n} \times x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 + \frac{1}{6} x^6$ &c. $= 1 - \overline{1-q}^{\frac{1}{2}}$ is the Logarithm of the ratio of 1 to $\overline{1-x}$, therefore $1 - \overline{1-x}^{\frac{1}{2}} = L$, and conſequently $\overline{1-x}^{\frac{1}{n}} = \overline{1-L}$, and $1 - x = \overline{1-L}^{\,n}$, wherefore by the ſame theorem $1 - x = \overline{1-L}^{\,2}$, will be equal to $1 - nL + \frac{1}{2} n^2 L^2 - \frac{1}{6} n^3 L^3 + \frac{1}{24} n^4 L^4 - \frac{1}{120} n^5 L^5 + \frac{1}{720} n^6 L^6 - \frac{1}{5040} n^7 L^7$, &c.

Whence $1 \pm x = 1 \pm nL + \frac{1}{2} n^2 L^2 \pm \frac{1}{6} n^3 L^3 + \frac{1}{24} {}^4 L^4 \pm \frac{1}{120} n^5 L^5$, &c. will be a general Theorem for finding the number from the Logarithm given, of any form whatſoever.

And becauſe $n = 1000000$, &c. in *Neper*'s form, $1 \pm x = 1 \pm L + \frac{1}{2} L^2 \pm \frac{1}{6} L^3 + \frac{1}{24} L^4 \pm \frac{1}{120} L^5$, &c. will give the number anſwering to any Logarithm in *Neper*'s form.

Hence any one term of the ratio, whereof L is the Logarithm being given, the other term will be readily found; for by putting $a$ for the leſſer of the two terms, and $b$ for the greater, $a \times 1 + L + \frac{1}{2} L^2 + \frac{1}{6} L^3 + \frac{1}{24} L^4 + \frac{1}{120} L^5$, &c. will give the greater, and $b \times 1 - L + \frac{1}{2} L^2 - \frac{1}{6} L^3 + \frac{1}{24} L^4 - \frac{1}{120} L^5$, &c. will give the leſſer, in *Neper*'s form.

Or,

$a \times 1 + nL + \frac{1}{2} n^2 L^2 + \frac{1}{6} n^3 L^3 + \frac{1}{24} n^4 L^4 + \frac{1}{120} n^5 L^5$, &c. $= b$ : and $a \times 1 - nL + \frac{1}{2} n^2 L^2 - \frac{1}{6} n^3 L^3 + \frac{1}{24} n^4 L^4 - \frac{1}{120} n^5 L^5$, &c. $= a$ in any form.

Let $d =$ the difference between the given Logarithm and the next neareſt tabular Logarithm; then will $a \times 1 + d + \frac{1}{2} d^2 + \frac{1}{6} d^3 + \frac{1}{24} d^4 + \frac{1}{120} d^5$, &c. $=$ N. Or, $b \times 1 - d + \frac{1}{2} d^2 - \frac{1}{6} d^3 + \frac{1}{24} d^4 - \frac{1}{120} d^5$, &c. $=$ N, the number required in *Neper*'s form. Or $a \times 1 + nd + \frac{1}{2} n^2 d^2 + \frac{1}{6} n^3 d^3 + \frac{1}{24} n^4 d^4 + \frac{1}{120} n^5 d^5$, &c. $=$ N. And $b \times 1 - nd + \frac{1}{2} n^2 d^2 - \frac{1}{6} n^3 d^3 + \frac{1}{24} n^4 d^4 - \frac{1}{120} n^5 d^5$, &c. $=$ N, the correſpondent number in Logarithms of any ſpecies.

Let it, for example, be required to find the number anſwering to 7.5713740280 in *Brigg*'s form.

Becauſe 7.5713710453, the neareſt tabular number to this is leſs than the given number, put 37271000, the number anſwering to it $= a$, and 0000029827, the difference between the two Logarithms $= d$, then will $n = 230258$, &c. multiplied by 37271000, the next neareſt number in the table, give 0000068683; which being increaſed by 1, and multiplied by 37271000, the next neareſt number in the table, will give 37271255.988 for the number anſwering to the given Logarithm: Or, if 0000068683 be multiplied by 37271000 $= a$, the product 255.998, added to $a = 37271000$ will give 37271255.988 for the number required, the ſame as before.

Hence, and from the general Theorem, may the number anſwering to any Logarithm be found, and conſequently an anti-logarithmic canon be conſtructed, ſhewing the natural numbers anſwering to every Logarithm; whence the numbers anſwering to any Logarithm might be found with the ſame eaſe as we find the Logarithm of any number in the common Logarithmic Cannon. Such a Cannon has been lately publiſhed by the ingenious Mr. *Dodſon*, which has rendered the Logarithmic Tables compleat.

Having thus ſhewn the method of conſtructing the Logarithmic Tables, we ſhall now proceed to ſhew how the Logarithmic ſines, tangents and ſecants are computed; but it will be requiſite previouſly to explain the manner how the natural ſine, tangent, ſecant, *&c.* of any arch may be found. In order to which let C (Plate IX. *Fig.* 19) be the center of a circle, $CA = CS = 1$ the radius, $AS = a$ any arch thereof, $Sr = s$ its right ſine; then will $Cr$, the co-ſine $= \sqrt{1 - ss}$, by the 47.1.

Let CM be another radius of the circle, infinitely near to CS; then will $Mn = \dot{s}$ be the fluxion of the ſine $Sr$, and the infinitely ſmall arch $MS = \dot{a}$, the fluxion of the arch AS. And becauſe the triangles $CSr$ and $MnS$ are ſimilar, it will be as $Cr : CS :: Mn : SM$; that is $\sqrt{1-ss} : 1 :: \dot{s} : \dot{a}$; wherefore $\dot{a} = \dfrac{\dot{s}}{\sqrt{1-ss}}$, that is, if the fluxion of the right-ſine be divided by the ſquare root of $1-ss$, the intregal, or flowing quantity of the quotient will be the arch itſelf.

The ſquare root of $1-ss$, being extracted will be $1 - \frac{1}{2}s^2 - \frac{1}{8}s^4 - \frac{1}{16}s^6$, *&c.* by which the fluxion $\dot{s}$ muſt be divided, which will give the fluxion of the arch SA, as follows,

$$\left.\begin{array}{l} 1 - \frac{1}{2}s^2 - \frac{1}{8}s^4 \\ - \frac{1}{16}s^6, \text{ \&c.} \end{array}\right\} \dot{s}\ \left(s + \frac{1}{2}s^2\dot{s} + \frac{3}{8}s^4\dot{s} + \frac{5}{16}s^6\dot{s}, \text{ \&c.}\right.$$

$$\begin{array}{r} \dot{s} - \frac{1}{2}s^2\dot{s} - \frac{1}{8}s^4\dot{s} - \frac{1}{16}s^6\dot{s}, \text{ \&c.} \\ \hline + \frac{1}{2}s^2\dot{s} + \frac{1}{8}s^4\dot{s} + \frac{1}{16}s^6\dot{s}, \text{ \&c.} \\ + \frac{1}{2}s^2\dot{s} - \frac{1}{4}s^4\dot{s} - \frac{1}{16}s^6\dot{s}, \text{ \&c.} \\ \hline + \frac{3}{8}s^4\dot{s} + \frac{1}{8}s^6\dot{s}, \text{ \&c.} \\ + \frac{3}{8}s^4\dot{s} - \frac{3}{16}s^6\dot{s}, \text{ \&c.} \\ \hline + \frac{5}{16}s^6\dot{s}, \text{ \&c.} \\ \hline \end{array}$$

Conſequently $s + \frac{1}{2}s^2\dot{s} + \frac{3}{8}s^4\dot{s} + \frac{5}{16}s^6\dot{s}$, &c. $= \dot{a}$, the fluxion of the arch, whoſe fluent or flowing quantity is $s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{112}s^7$, &c.

Therefore $a = s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{112}s^7$, &c. or $a =$ $s + \frac{1 \times 1}{2 \times 3}s^3 + \frac{1 \times 1 \times 3 \times 3}{2 \times 3 \times 4 \times 5}s^5 + \frac{1 \times 1 \times 3 \times 3 \times 5 \times 5}{2 \times 3 \times 4 \times 5 \times 6 \times 7}s^7$, &c.

Hence we may find the length of any arch of a circle, the right-ſine of that arch being given.

And becauſe the radius or ſemidiameter of a circle is equal to the ſide of a hexagon inſcribed in a circle (*Cor.* 1. *Prop.* 15. *Book* 4.) or chord of 60 degrees, therefore the ſine of 30 degrees will be equal to half the radius: And conſequently when the radius of a circle is given, the ſine of 30 degrees is given alſo.

Let it therefore be required to find the length of the arch of thirty degrees, to ten decimal places, which is abundantly ſufficient for any common purpoſe, the radius being = 1.

Becauſe the radius = 1, the ſine of 30 deg. $= s$, will be $= \frac{1}{2}$, and conſequently its ſquare $\frac{1}{4}$, its cube $\frac{1}{8}$, &c. And as to multiply by $\frac{1}{4}$, and divide by 4, is the ſame thing, it follows that if the right-ſine $= \frac{1}{2} = .5$ be divided by 4, and the ſeveral quotients reſulting from thence be multiplied by 1, $\frac{1}{6}$, $\frac{3}{40}$, &c. the ſum of theſe products will be the length of the arch required.

| | | | | | |
|---|---|---|---|---|---|
| $s$ | $=$ .5000, 0000, 000 | | | .5000, 0000, 000 | |
| $s^3$ | $=$ 1250, 0000, 000 | its | $\frac{1}{6}$ | 208, 3333, 333 | + |
| $s^5$ | $=$ 312, 5000, 000 | its | $\frac{3}{40}$ | 23, 4375, 000 | |
| $s^7$ | $=$ 78, 1250, 000 | its | $\frac{5}{112}$ | 3, 4877, 232 | + |
| $s^9$ | $=$ 19, 5312, 500 | its | $\frac{35}{1152}$ | 5933, 974 | − |
| $s^{11}$ | $=$ 4, 8828, 125 | its | $\frac{63}{2816}$ | 1092, 391 | − |
| $s^{13}$ | $=$ 1, 2207, 031 | its | $\frac{231}{13312}$ | 211, 826 | − |
| $s^{15}$ | $=$ 3851, 758 | its | $\frac{143}{10240}$ | 42, 617 | + |
| $s^{17}$ | $=$ 762, 939 | its | $\frac{6435}{57058}$, &c. | 8, 813 | + |
| $s^{19}$ | $=$ 190, 735 | its | $\frac{12155}{124518}$, &c. | 1, 863 | − |
| $s^{21}$ | $=$ 47, 684 | its | $\frac{46189}{550502}$, &c. | 400 | + |
| $s^{23}$ | $=$ 11, 921 | its | $\frac{67899}{928514}$, &c. | 87 | + |
| $s^{25}$ | $=$ 2, 980 | its | $\frac{3501}{557108}$, &c. | 19 | + |
| $s^{27}$ | $=$ 745 | its | $\frac{2244}{391090}$, &c. | 4 | + |

The length of the arch of 30 deg. = .5235, 9877, 559

If this arch be multiplied by 6, we ſhall have the length of the arch of the ſemicircle when the radius is 1, or the length of the whole circumference when the diameter is 1.

$$\begin{array}{r} .5235,9877,559 \\ 6 \\ \hline 3.1415,9265,354 \end{array}$$

The abſolute neceſſity of finding, as near as poſſible, the proportion between the diameter and circumference of a circle, for exactly true it never can be found, has engaged ſome of the greateſt Mathematicians to approximate as near as poſſible to the truth. *Van Ceulen* carried it to 32 decimal places, which he ordered to be engraved on his Tomb-ſtone, as a memorial of ſo great a work. His numbers are as follows,

If the diameter be 1.0000000, *&c.* the circumference will be 3.141592,653589,793238,462643,383279,50288 +.

Since *Van Ceulen*, the indefatigable Mr. *Abraham Sharp*, has carried it to twice the number of places, and ſhewn us that if the diameter be 1.0000000, *&c.* the circumference will be 3.141592,653589,793238,462643,383279, 502884,197169,399375,105820,974944,592307,816405, 2 +.

And the learned Mr. *John Machin*, has carried it to 100 places, which is a degree of exactneſs far exceeding all imagination. And is as follows.

If the diameter of a circle be 1,000, *&c.* the circumference will be 3.141592,653589,793238,462643,383279,502884, 197169,399375,105820,974944,592307,816405, 286208, 998628,034825,342117,0679 + of the ſame parts.

Becauſe $a = s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{11}s^7$, *&c.* Therefore,

$$s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5040}a^7, \text{ \&c.}$$

Or,

$$s = a - \frac{1}{2\times3}a^3 + \frac{1}{2\times3\times4\times5}a^5 - \frac{1}{2\times3\times4\times5\times6}a^7, \text{ \&c.}$$

For put $s = Aa + Ba^3 + Ca^5$, *&c.*

$$\left.\begin{array}{ll} \text{Then } \frac{1}{6}s^3 = + \frac{1}{6}A^3a^3 + \frac{1}{2}A^2Ba^5, \text{ \&c.} \\ \text{And } \frac{3}{40}s^5 = \qquad\qquad + \frac{3}{40}A^5a^5, \text{ \&c.} \end{array}\right\} = a$$

Conſequently $\mathrm{A}a = a$, and $\mathrm{A} = 1$; alſo $\mathrm{B} + \frac{1}{6}\mathrm{A}^3 = 0$, and $\mathrm{B} = -\frac{1}{6}$. Alſo $\mathrm{C} + \frac{1}{2}\mathrm{A}^2\mathrm{B} + \frac{3}{40}\mathrm{A}^5 = 0$. And $\mathrm{C} = -\frac{1}{2}\mathrm{A}^2\mathrm{B} - \frac{3}{40}\mathrm{A}^5 = -\frac{1}{2} \times -\frac{1}{6} - \frac{3}{40} = +\frac{1}{120}$; wherefore $\mathrm{A} = 1$, $\mathrm{B} = -\frac{1}{6}$, $\mathrm{C} = +\frac{1}{120}$ &c. wherefore $s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. *Q. E. D.*

Let it be required to find the natural ſine of 10 degrees. Becauſe the length of the ſemi-circumference is 3.1415, 9265, 354, therefore the length of 10 deg. or $\frac{1}{18}$ will be .1745, 3292, 5196.

Put therefore $a =$ ———————— 1745, 3292, 519
Then will $-\frac{1}{6}a^3 =$ ———————— 8, 8609, 615
And $a - \frac{1}{6}a^3 =$ ———————— 1736, 4682, 904
Again $+\frac{1}{120}a^5 =$ ———————— 134, 960
And $a - \frac{1}{6}a^3 + \frac{1}{120}a^5 =$ ———————— 1736, 4817, 864
Alſo $-\frac{1}{5040}a^7 =$ — 97
And $a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5040}a^7 =$ 1736, 4817, 767

Therefore $s =$ .1736, 4817, 767, the ſign of 10 degrees required.

The ſine of an arch being given, its co-ſine may be eaſy found; for becauſe $\mathrm{CS}q = \mathrm{SR}q + \mathrm{CR}q$ (by 47.1) ſee Plate IX. *Fig.* 3. therefore $\sqrt{\mathrm{CS}q - \mathrm{SR}q} = \mathrm{CR}$. That is, if from the ſquare of the radius, be taken the ſquare of the right ſine, the ſquare root of the remainder will be the co-ſine required.

Therefore if from the ſquare of radius $= 1$, be taken the ſquare root of the right-ſine $= a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. the ſquare root of the remainder will be the co-ſine required.

Thus,

$$\begin{array}{rl}
s = & a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\
s = & a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\
\hline
 & a^2 - \frac{1}{6}a^4 + \frac{1}{120}a^6, \text{ \&c.} \\
 & \quad - \frac{1}{6}a^4 + \frac{1}{36}a^6, \text{ \&c.} \\
 & \quad\quad + \frac{1}{120}a^6, \text{ \&c.} \\
\hline
\end{array}$$

$ss = a^2 - \frac{1}{3}a^4 + \frac{2}{45}a^6$, &c. which taken from the ſquare of radius $= 1$, leaves $1 - a^2 + \frac{1}{3}a^4 - \frac{2}{45}a^6$, &c. the ſquare root of which will be the co-ſine required.

$$\begin{array}{rl}
 & 1 - aa + \frac{1}{3}a^4 - \frac{2}{45}a^6, \text{ \&c. } (1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6, \text{ \&c.} \\
 & 1 \\
\hline
2 - \frac{1}{2}aa) & - aa + \frac{1}{3}a^4 \\
 & - aa + \frac{1}{4}a^4 \\
\hline
2 - aa, \text{ \&c.}) & + \frac{1}{12}a^4 - \frac{2}{45}a^6, \text{ \&c.} \\
 & + \frac{1}{12}a^4 - \frac{1}{24}a^6, \text{ \&c.} \\
\hline
2 - aa, \text{ \&c.}) & - \frac{1}{360}a^6, \text{ \&c.} \\
 & - \frac{1}{360}a^5, \text{ \&c.} \\
\hline
\end{array}$$

Wherefore the co-ſine $= 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^5$, &c.

Or the co-ſine $= 1 - \frac{1}{1 \times 2}a^2 + \frac{1}{1 \times 2 \times 3 \times 4}a^4 - \frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6}a^6$, &c.

The ſines being thus obtained, the tangents and ſecants are eaſily found as follows.

For becauſe the triangles CRS, CAT, (Plate IX. *Fig.* 3.) are ſimilar, it will be,

As CR, the co-ſine, is to RS, the right-ſine; ſo is CA, the radius, to AT, the tangent.

And becauſe the radius $= 1$, therefore the quotient of the right-ſine divided by the co-ſine will give the tangent of the ſame arch.

Hence the tangent ſeries are eaſily found; for if $a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. the right-ſine, be divided by $1 - \frac{1}{2}aa + \frac{1}{24}a^4$, &c. the co-ſine, the quotient will give the tangent required, as in the following example.

$$\begin{array}{l} 1-\frac{1}{2}aa+\frac{1}{24}a^4,\ \&c.)\ a-\frac{1}{6}a^3+\frac{1}{120}a^5,\ \&c.\ (a+\frac{1}{3}a^3+\frac{2}{15}a^5,\ \&c. \\ \qquad\qquad\qquad\quad \underline{a-\frac{1}{2}a^2+\frac{1}{24}a^5,\ \&c.} \\ \qquad\qquad\qquad\qquad +\frac{1}{3}a^3+\frac{1}{30}a^5,\ \&c. \\ \qquad\qquad\qquad\qquad \underline{+\frac{1}{3}a^3-\frac{1}{6}a^5,\ \&c.} \\ \qquad\qquad\qquad\qquad\qquad\quad +\frac{2}{15}a^5,\ \&c. \end{array}$$

Hence if $a =$ the length of any arch, the correſpondent tangent will be $a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{17}{315}a^7+\frac{62}{2835}a^9+\frac{1382}{155925}a^{11}$, &c.

Again, becauſe the triangles ACT, GBC, are ſimilar, it will be, as AT, the tangent, to AC, radius; ſo is CB, radius, to BG, the co-tangent.

Hence the radius is a mean proportional between the tangent and co-tangent of an arch, and becauſe the radius $=1$, if 1 be divided by $a+\frac{1}{3}a^3+\frac{2}{15}a^5$, &c. the tangent, the quotient $\frac{1}{a}-\frac{1}{3}a-\frac{1}{45}a^3-\frac{2}{945}a^5-\frac{1}{4725}a^7-\frac{2}{93555}a^9$, &c. will be the ſeries for finding the co-tangent, from the arch firſt given.

Again, becauſe the triangles CRS, CAT, are ſimilar, it will be as CR, the co-ſine, to CS, the radius; ſo is CA, the radius, to CT, the ſecant.

Hence the radius is a mean proportional between the ſecant and co-ſine of an arch.

And therefore if 1 be divided by $1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{72}a^6$, &c. the co-ſine, the quotient $1+\frac{1}{2}a^2+\frac{5}{24}a^4+\frac{71}{720}a^6+\frac{277}{8064}a^8+\frac{50521}{3628800}a^{10}$, &c. will be the ſeries for finding the ſecant from the arch firſt given.

A cannon of natural ſines, tangents, and ſecants being conſtructed according to the foregoing examples, it will be eaſy to find the correſponding artificial ſines, tangents, and ſecants; for the Logarithms of the former will be the latter.

But the Logarithmic ſine, tangent, or ſecant of any arch may be found, independantly of the tables of Logarithms; for by putting $s$ for the ſine of any arch, and $q=\frac{s-1}{s+1}$ we ſhall have $\frac{1}{n}\times q+\frac{1}{3}q^3+\frac{1}{5}q^5+\frac{1}{7}q^7$, &c. for the Logarithmic ſine of the ſame arch, and if inſtead of $s$, in the former ſeries, we put $t$ for the tangent, or $c$ for the ſecant of any arch, the ſame ſeries will produce the Logarithmic tangent or ſecant of the given arch.

Or if inſtead of $s$, the right-ſine, in the above ſeries we put $a$, for the length of the arch itſelf then will $\frac{1}{n}\times n-\frac{1}{2}a^2-\frac{1}{12}a^4-\frac{1}{45}a^6-\frac{17}{2520}a^8-\frac{31}{14175}a^{10}$, &c. be the

Logarithmic co-ſine of the ſame arch, in *Brigg's* form, the radius being 10.0000000000, and $n = 2.\,3025,8509,29$.

We have already ſhewn that if the radius of a circle be put equal to 1, the length of the arch of 10 degrees will be equal to 1745, 3292, 519, of the ſame parts; put this therefore equal to $a$.

Then will $\frac{1}{2}a^2 =$ ——————— 0152, 3087, 10
And $\frac{1}{12}a^4 =$ ——————— 7732, 65
Alſo $\frac{1}{45}a^5 =$ ——————— 62, 81
And $\frac{17}{2520}a^8 =$ ——————— 58
Therefore $\frac{1}{2}a^2 + \frac{1}{12}a^4 + \frac{1}{45}a^6 + \frac{17}{2520}a^8 =$ 0153, 0883, 14

This value being taken from $n = 2.\,3025,8509,29$, will leave 2, 2872, 7625, 15, which being multiplied by 4342, 9448, 190, will give the fractional part of the Logarithmic ſine of 80 degrees = .9933, 5145, 89, to which prefixing 9 for an index, becauſe the radius is equal to 10, will give 9.9933, 5145, 89, for the Logarithmic ſine of 80 degrees in *Brigg's* form; and after the ſame manner may the ſine of any arch be found independant of the Logarithmic tables, and conſequently the whole table of ſines conſtructed.

Having thus computed a table of Logarithmic ſines, the Logarithmic tangents and ſecants may be had by common addition and ſubtraction only. For,

1. As the co-ſine of an arch, is its right ſine; ſo is the radius, to the tangent of the ſame arch.

Therefore if to the right-ſine of an arch we add the radius, and from that ſum ſubtract the co-ſine of the ſame arch, the remainder will be the tangent required.

Thus, To the Log. ſine of 80 degrees = 9.99335,14589
Add the radius ——————— = 10. 0000000000
19.99335,14589
And ſubtract the co-ſine of 80 degrees = 9.23967,02300
The remainder is the Log. tang. of 80 deg. = 10.75368,12289

2. As the co-ſine, is to radius; ſo is radius to the ſecant of the ſame arch.

Therefore, from twice radius ſubtract the co-ſine, and the remainder will give the ſecant of the ſame arch.

Thus, from twice radius ——————— = 20. 0000000000
Take the co-ſine of 10 degrees — = 9.99335,14589
The remainder is the ſecant of 10 deg. = 10.00664,85411

Hence it appears that if the Logarithmic ſine and co-ſine of an arch be known, its tangent and ſecant may be readily found; and conſequently the whole cannon eaſily computed.

The ingenious Mr. *Sympſon*, in his Trigonometry, page 47 & ſeq. has given us a very compendious method of finding the Logarithms of large numbers, one from another, by addition and ſubtraction only, which is as follows.

1. Let A, B, C, denote any three numbers in arithmetical progreſſion, not leſs than 10000 each, whereof the common difference is 100.

2. From twice the Logarithm of B, ſubtract the ſum of the Logarithms of A and C, and let the remainder be divided by 10000.

3. Multiply the quotient by 49.5, and to the product add $\frac{1}{100}$ part of the difference of the Logarithms of A and B; then the ſum will be the exceſs of the Logarithm of $A + 1$ above that of A.

4. From this exceſs let the quotient (found by Rule 2.) be continually ſubtracted, that is, firſt from the exceſs itſelf, then from the remainder, then from the next remainder, *&c. &c.*

5. To the Logarithm of A add the ſaid exceſs, and to the ſum add the firſt of the remainders; to the laſt ſum add the next remainder, *&c. &c.* then the ſeveral ſums, thus ariſing, will exhibit the Logarithms of $A + 1$, $A + 2$, $A + 3$, *&c.* reſpectively.

Thus let it be propoſed to find the Logarithms of all the whole numbers between 17900 and 18100; thoſe of the two extremes 17900 and 18100, and that of the mean (18000) being given.

$$\text{Then the Logarithm of } \left\{\begin{matrix} A \\ B \\ C \end{matrix}\right\} \text{ being equal to } \left\{\begin{matrix} 4.252853031 \\ 4.255272505 \\ 4.257678575 \end{matrix}\right\}$$

$$\text{we ſhall have } \frac{2\ \text{Log. B} - \text{Log. A} - \text{Log. C}}{10000} = .000000000134$$

(ſee Rule 2.) which multiplied by 49.5, and the product added to $\frac{\text{Log. B} - \text{Log. A}}{100}$ gives .00002426107 for the exceſs of the Logarithm of $A + 1$ above that of A (by Rule 3.) From whence the work, being continued according to Rule 4 and 5, will ſtand as follows.

| | | | | | |
|---|---|---|---|---|---|
| 0000024 | 26107 | excefs | 4.25 | 2853031 | Log. of 17900. |
| | 134 | | | 2426107 | excefs |
| | 25973 | *1ſt* Rem. | | 287729207 | Log. of 17901. |
| | 134 | | | 2425973 | |
| | 25839 | *2d* Rem. | | 290155280 | Log. of 17902. |
| | 134 | | | 2425839 | |
| | 25705 | *3d* Rem. | | 292581019 | Log. of 17903. |
| | 134 | | | 2425705 | |
| | 25571 | *4th* Rem. | | 295006724 | Log. of 17904. |
| | 134 | | | 2425571 | |
| | 25437 | *5th* Rem. | | 297432295 | Log. of 17905. |
| | 134 | | | 2425437 | |
| | 25303 | *6th* Rem. | | 299857732 | Log. of 17906. |
| | 134 | | | 2425303 | |
| | 25169 | *7th* Rem. | | 302283035 | Log. of 17907. |
| | 134 | | | 2425169 | |
| | 25035 | *8th* Rem. | | 304708204 | Log. of 17908. |
| | 134 | | | 2425035 | |
| | 24901 | *9th* Rem. | | 307133239 | |
| | 134 | | | 2424901 | |
| | 24767 | *10th* Rem. | | 309558140 | Log. of 17909. |
| | &c. | &c. | | &c. | &c. |

*Note*, The Longarithms found by this method, in numbers between 10000 and 20000, are true to 8 or 9 places of figures. Thoſe of numbers between 20000 and 50000 err only in the 9th or 10th place; and thoſe above 50000 are true to ten places at leaſt.

Having ſufficiently explained the manner of conſtructing the tables of Logarithms, it remains that we ſhew the uſe of thoſe tables.

1. We having already obſerved that addition of Logarithms anſwers to multiplication of natural numbers, and ſubſtraction to diviſion. Thus if it were required to multiply 8.5 by 10, the operation will be as follows,

| | |
|---|---|
| To the Log. of 8.5. ———————— | = 0.9294189 |
| Add the Log. of 10 ———————— | = 1.0000000 |
| The Sum ———————————— | 1.9294189 |

Which is the Log. of the product required, and the number correſponding thereto is 85.

Or if it were required to divide 9712.8 by 456, it will ſtand thus,

From the Log. of 9712.8 ——— = 3.9873444
Take the Log. of 456 ——— = 2.6589648
Remains ——— 1.3283796

Which is the Log. of the quotient required, and whoſe correſponding number is 21.3.

*To find the Complement of a Logarithm.*

Begin at the left-hand, and ſet down the difference between each figure and 9, except the firſt, which muſt be ſubtracted from ten. Thus you will find the complement of the Log. of 456, *viz.* of 2.6589648 to be 7.3410352.

2. *To perform the rule of Proportion by the Logarithms.*

Add the Logarithms of the ſecond and third terms together, and ſubtract the Log. of the firſt from that ſum; the remainder will be the Log. of the fourth term or number ſought. Or inſtead of ſubtracting the Log. of the firſt term, add its complement, and the reſult will be the ſame: But obſerve that in adding a complement you muſt always cancel 1 in the tens place of the index, for every complement ſo added.

Thus, let it be required to find a fourth number in geometrical proportion to the three following, *viz.* 4, 9, 12.

To the Log. of 9 ——— = 0.9542425
Add the Log. of 12 ——— = 1.0791812
And from their ſum ——— = 2.0334237
Take the Log. of 4 ——— = 0.6020600
The remainder is the Log. of 27 ——— = 1.4313637.
So that the fourth number required is 27.

Or thus,

Complement of the Log. of 4 is ——— 9.3979400
Log. of ——— 9 ——— 0.9542425
Log. of ——— 12 ——— 1.0791812
Sum is Log. of 27 ——— ~~1~~1.4313637

The ſame as before, except in the index, where 1 in the place of tens muſt be cancelled, as we before obſerved.

*Example* 2. Suppoſe 100*l.* in one year, or 365 days, gain 6*l.* intereſt; what will 5173*l.* gain in 321 days?

*Anſwer,* 272.9643*l.* or 272*l.* 19*s.* 3*d.* ½ nearly.

Comp.

Plate IX. *Facing Pag. 380.*

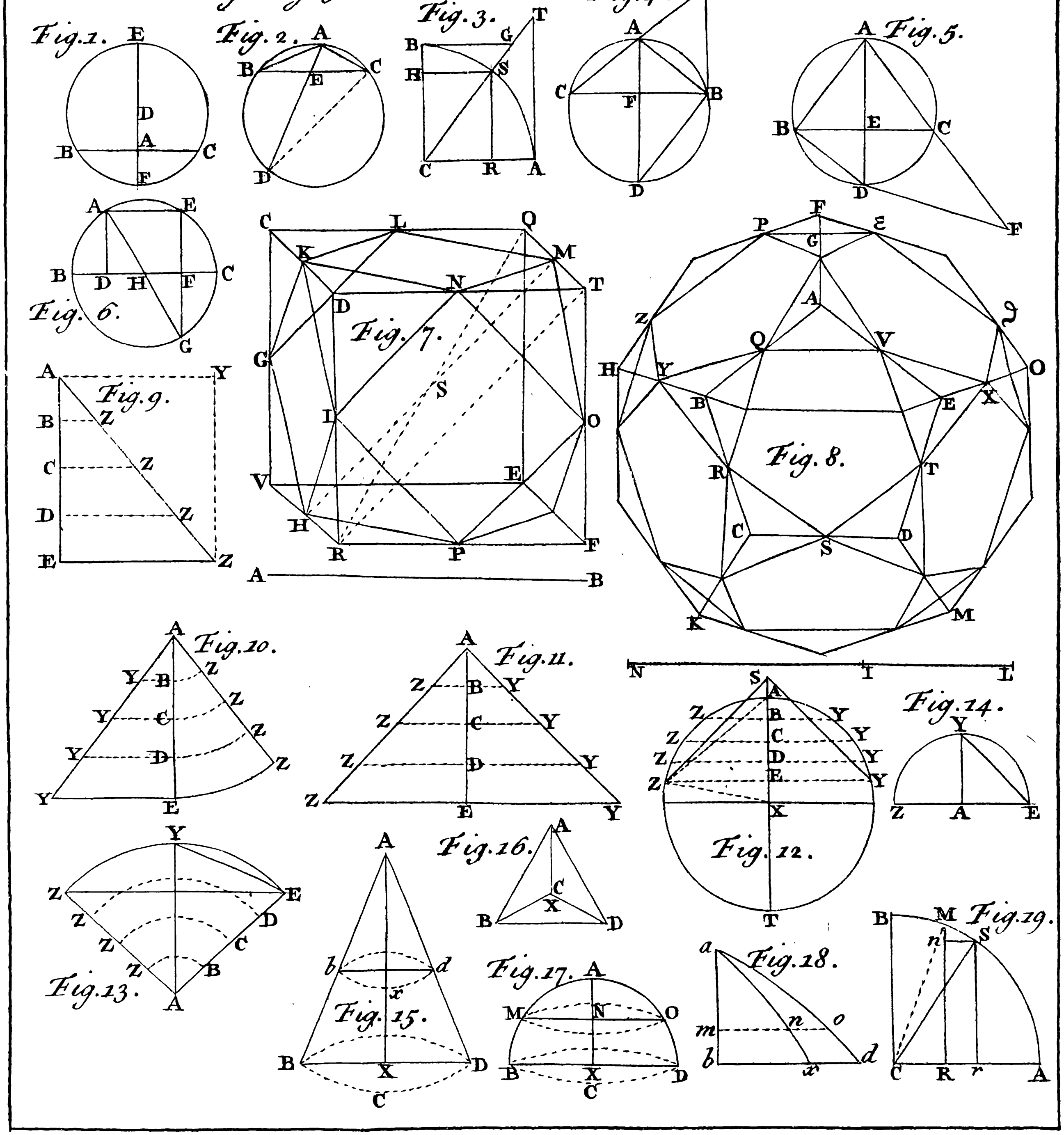
Fig. 1.
E
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Fig. 2.
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Fig. 4.
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Fig. 5.
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Fig. 6.
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Fig. 7.
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Fig. 8.
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Fig. 9.
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Fig. 10.
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Fig. 11.
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Fig. 12.
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Fig. 13.
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Fig. 14.
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Fig. 15.
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Fig. 16.
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Fig. 17.
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Fig. 18.
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Fig. 19.
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M
n
S
C
R
r
A

| | | |
|---|---|---|
| Comp. of the Log. of | 100 | 8.0000000 |
| Comp, of the Log. of | 365 | 7.4377071 |
| Log. of | 6 | 0.7781513 |
| Log. of | 5173 | 3.7137425 |
| Log. of | 321 | 2.5065050 |
| Log. of | 272.9643 | $\cancel{2}$2.4361059 |

In the above operation, 2 in the place of tens in the index is cancelled, becauſe two complements are added.

### 3. *To raiſe Powers by Logarithms.*

Multiply the Logarithm of the given number by the index of the required power, the product will be the Logarithm of the power ſought.

*Example.* Let it be required to find the cube of 32 by the Logarithms.

The Log. of 32 is ——— 1.5051500
Which multiply by the index of the power ——— 3
The Log. of 32768 ——— 4.5154500
Which is the cube of 32 required.

*Example* 2. Let it be required to find the cube of 0.009 by the Logarithms.

The Log. of 0.009 is (*ſee page* 368) ——— 3.9542425
Which multiply by ——— 3
Which is the Log. of 0.000000729 ——— 7.8627275

*N.B.* When the Log. has a negative index, as in the above example, whaterever is carried from the decimal part of the Log. muſt be conſidered as affiirmative, and ſubtracted from the multiplication of the negative index; for the decimal part of every Log. is affirmative, whether the whole Log. taken together be ſo or not. Thus in the above example, the negative index 3 being multiplied by 3, produces 9, from which the 2 carried from the decimal part being ſubtracted, gives — 7 for the index of the Log. required.

*Example* 3. Let the 6th power of 0.0032 be required.
The Log. of 0.0032 is ——— 3.50515500
Which multiplied by ——— 6
The product is ——— 15.03093000
Which is the Log. of ——— 0.000000000000000010739

4. *To*

### 4. *To extract the Roots of Powers by the Logarithms.*

Divide the Log. of the number by the index of the power, the quotient is the index of the root ſought.

*Example,* Let it be required to extract the cube root of 6751269.

The Log. of 6751269 is 6.8293854, which being divided by 3, the index of the power, gives 2.2764618. the Log. of 189, the cube root ſought.

*Example* 2. Let it be required to extract the root of the 6th power of 0.00000,00000,00000,00000,00000,00000,63 by the Logarithms.

The Log. of 0.00000,00000,00000,00000,00000,00000, 63 is — 31.8061800, which being divided by 6, the index of the power gives — 5.3010300, which is the Log. of 0.00002, the root of the ſixth power required.

And after the ſame manner may the root of any other power be extracted.

### 5. *To find mean proportionals between any two numbers.*

Subtract the Log. of the leaſt term from the Log. of the greater, and divide the remainder by a number which exceeds by one the number of means ſought; then add the quotient to the Log. of the leſſer term, or ſubtract it from the Log. of the greateſt, continually, and you will have the Logarithms of all the mean proportionals required.

*Example.* Let it be required to find three mean proportionals between 106 and 100.

Log. of 106————2.0253059  
Log. of 100————2.0000000  
Divide by ———— 4)0.0253059(0.0063264.75  
Log. of the firſt mean————2.0000000  
Exceſs added————0.0063264.75  
Log. of the firſt mean 101.4673846 =2.0063264.75  
Exceſs added————0.0063264.75  
Log. of the ſecond = 102.9563014 =2.0126529.50  
Exceſs added ————0.0063264.75  
Log. of the third—— 104.4670483=2.0189794.25  
Exceſs added————0.0063264.75  
Log, of the fourth——106————2.0253059

### *The uſe of the tables of Logarithmic ſines, tangents and ſecants.*

1. To find the ſine, tangent and ſecant of any arch to ninety degrees.

Find the number of degrees at the head or foot of the table; and the minutes in the left or right-hand column, and in the common angle of meeting, formed by the title placed at the head or foot of the column, and the minutes, you will find the ſine, tangent or ſecant required.

But if the given arch contains any parts of a minute, intermediate to thoſe found in the tables; take the difference between the ſines, *&c.* of the given degrees and minutes, and of the minute next greater.

Then as 1 minute is to that difference, ſo is the given intermediate part of a minute in decimals to a fourth proportional, Which being added to the ſine, *&c.* firſt found will give the ſine, *&c.* required.

*Example.* The Log. ſine of 1°. 48′. 28″. 12‴, is required.

The Log. ſine of 1°. 49′. is ——— 8.5010798
The Log. ſine of 1°. 48′. is ——— 8.4970784
Their difference is ——— 40014

Then as 1 : 40014 : : 0′.47 (=28″. 12‴.): 18807 nearly.

Therefore to the Log. ſine of 1°. 48′. ——— 8.4970784
Add the difference found ——— 40014
The ſum is the Log. ſine of 1°. 48′. 28″. 12‴ — 8.5010798

### 2. *To find the arch anſwering to any given ſine, tangent, or ſecant.*

Take the next leſs found in the tables, and ſubtract it from that given, obſerving the degrees and minutes anſwering to it, and divide this remainder by the difference between it and the next greater, adding as many cyphers as are neceſſary to the dividend, and the quotient will be the decimal part of a minute to be added to the degrees and minutes before found.

*Example* 1. Suppoſe it were required to find the arch anſwering to the Log. ſine ——— 9.8393859
The next leſs is the Log. ſine of 43°. 41′. — 9.8392719
The difference is ——— 1140

Alſo the difference between the Log. ſine of 43°. 41′, and 43°. 42′. is 1322.

Then 1322)1140.000(0′. 862 = 51″. 43‴, which being added to 43°. 41′, gives 43°. 41′. 51″. 43‴, for the arch required.

*Example* 2. Let it be required to find the arch correſponding to the Log. tangent ———— 9.6766687
The tang. of 25°. 24′, the next leſs is ———— 9.6765426
The difference is ———— 1261

Alſo the difference between the Log. tangent of 25°. 24′. and the Log. tangent of 25°. 25′. is 3260.

Then, $3260)1261.0000(0'.\ 387 = 23''.\ 13'''$.

Therefore 25°. 24′. 23″. 13‴, is the arch anſwering to the given tangent.

# *F I N I S.*